NWT Apprenticeship Support Materials

Math

Reading Comprehension

Science

* Module 1 – Foundations
* Module 2 – Patterns and Relations
* Module 3 – Variables and Equations
* Module 4 – Measuring Time, Shapes and Space
* *Module 5 – Special Topics*

PARTNERS

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Northwest Territories Education, Culture and Employment

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Table of Contents

Module 5 – Special Topics

Introduction[1]

Module 5 math covers special topics used in electrical trades and electronic technologies. These topics are also useful for studies in engineering and science. Individuals taking the level five trades entrance exam will need to understand the topics covered in Math – Module Five and Science – Module Three. The content covered in Mathematics Modules 1-4 is a pre-requisite for level 5 math.

Organization of topics

The emphasis in trades is on using mathematics to solve practical problems quickly and correctly. Each topic in this curriculum guide includes:

1. Background and theory
2. Examples with explanations
3. Practice Exam Questions with answers and explanations

This curriculum guide outlines competencies, but does not provide detailed lessons, as for example in a textbook or college mathematics course. If you need more instruction on a particular competency, you may find these resources helpful. If you want to build up speed as well as accuracy in a competency you will find these resources helpful as a source of additional practice questions. Need to know information for the trades entrance exam is singled out for your attention by the use of text boxes and bold type.

Examples are the focus

In this curriculum guide, examples with explanations are the primary tool used for review. Background for each competency is also given with a brief overview of what you need to know. Before any examples are given, the main ideas in each topic are explained and "need to know information" is summarized in rules and definitions.

1 A detailed introduction with sections on self-assessment and study tips is provided with Math – Module 1 – Foundations. Please review this material to supplement the brief overview provided here.

Introduction

Please Note:

When you work on an example, cover the text below with the laminated card provided so that you don't see answers and explanations prematurely.

You may want to skip the background given on a topic and go right to the examples to see how well you do. You can always go back to the theory if you find you need it. In addition, a set of practice exam questions accompanies this learning guide to enable you to assess yourself, decide what you need to study, and practice for the exam.

Unit 1

Equations and Patterns

Topic 1 – Functions

In Modules 2 and 3 of the Math curriculum you were introduced to patterns, relations and equations. In the trades, equations based on measurements produce formulas and **formulas are functions**. Review the material on finding values in a series from Module 2 for this topic. Formulas are developed from data that show useful relationships between variables that we are interested in.

For example, the formula V=IR (Ohm's law) describes the relationship between voltage, current, and resistance in an electrical circuit.[2] These three variables are functions of each other. The value of each variable is determined by the values of the other two. When we know any two values, the third can be calculated by using the formula. Experiments with current, voltage and resistance produce data that can be recorded in a table. The values in the table reveal a rule of correspondence between V, I and R.

A function is a set of ordered pairs that is determined by a pattern.

Study the following table to discover this rule.

V(volts)	I (Amperes)	R (Ohms)
6	3	2
10	5	2
15	10	1.5
8	6	?
?	5	10
12	?	6

2 See Science – Module 3 – Special Topics for applications of Ohm's law.

You can see that V = IR. In words, "voltage equals the product of current and resistance". Notice that the formula can be rearranged to define I in terms of R and V: $I = \frac{V}{R}$, and R in terms of V and I: $R = \frac{V}{I}$. From this table we can write a formula and use it to make predictions. For example, if we know that V = 10 volts, and I = 10 amperes, then the resistance in the circuit will be 1 Ohm. The question marks in the table can be filled in by solving the equations for each row in the table. For example, in row four, 8 = 6 x ?, and ? = $\frac{8}{6}$ = $\frac{4}{3}$ = 1.33 Ohms.

You should be able to use the formula to fill in the blanks for row five = 50 volts, and row six = 2 amps.

Examples

1. What formula describes the relationship between the variables in the following table:

input (x)	output f(x)=y
10	5
3	1.5
1	.5
200	100

This table has a list of ordered pairs that shows a functional relationship between an input number and an output number. Notice that the input numbers are not listed by size. **The relationship does not depend on the order of the input numbers**. The expression f(x) means "function of x". The equation y = f(x) says "the value of y is a function of (i.e. depends upon) the value of x". The value of x (an input number) determines the value of y (an output number). In the trades, numbers used in measurement will appear in formulas. These numbers include integers, rational numbers (fractions), and irrational numbers.

In this example the output number, aka the **range** of the function is always one half the size of the input number, aka the **domain** of the function. We divide any input number by 2 to get the corresponding output number. This is the rule of correspondence between an input number and an output number. The equation $y = \frac{x}{2}$ describes the relationship. For example, (10,5) is an ordered pair that satisfies the relationship because $5 = \frac{10}{2}$.

This function can also be described algebraically with an ordered pair that uses variables: $(x, \frac{x}{2})$. Only the numbers that satisfy this description of ordered pairs will be solutions to the equation $y = \frac{x}{2}$. (20, 10) is a solution because $10 = \frac{20}{2}$, but (20, 5) is not because 20 ÷ 2 ↑ 5. Every ordered pair that satisfies $(x, \frac{x}{2})$ will be on a graph of the function, and the graph will only go through points that satisfy the equation of the function. The following table illustrates how to choose input numbers that are convenient for sketching the graph of $y = \frac{x}{2}$. Any number can be used as an input number unless we are told otherwise. Input numbers that result in division by zero are not part of the domain of a function. When this situation occurs, we say that the function is not defined for such an input value.[3]

[3] For example, the tangent of 90° is not defined because the cosine of 90° is 0, and the tangent would be found by dividing the sine = 1 by the cosine = 0.

Unit 1 – Equations and Patterns

Topic 1 – Functions

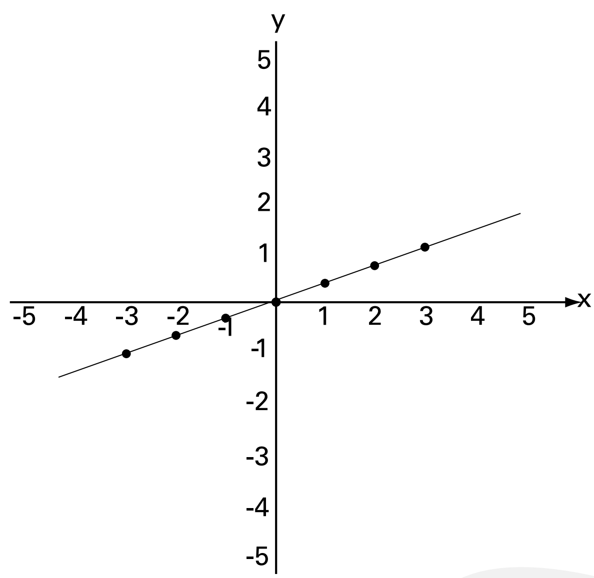
x	$y = \frac{x}{2}$
-3	-1.5
-2	-1
-1	-.5
-.5	-.25
0	0
.5	.25
1	.5
2	1
3	1.5

Ordered pair: (input (x), output (Y))

In a function, there is only one output number (y) for each input number (x). However, an output number may correspond to more than one input number.

Input numbers are the x coordinates on a graph, and output numbers are the y coordinates.

Here you see the line that connects the points given in the table for $y = \frac{x}{2}$.



The line goes through all of the points in the function $y = \frac{x}{2}$, not only those selected for inclusion in a table. It can be extended indefinitely in both directions. This is a linear function.[4] Any number studied in this curriculum can be used as an input number, and every point on the line can be identified by an ordered pair of numbers that is a solution for the function $f(x) = \frac{x}{2}$. This function is an example of a **continuous function** because it will have solutions that can be graphed in a continuous smooth line. There are no breaks in the graph.

4 See Math – Module 3 – Variables and Equations, Topic 1B: Solving Linear Equations.

Unit 1 – Equations and Patterns

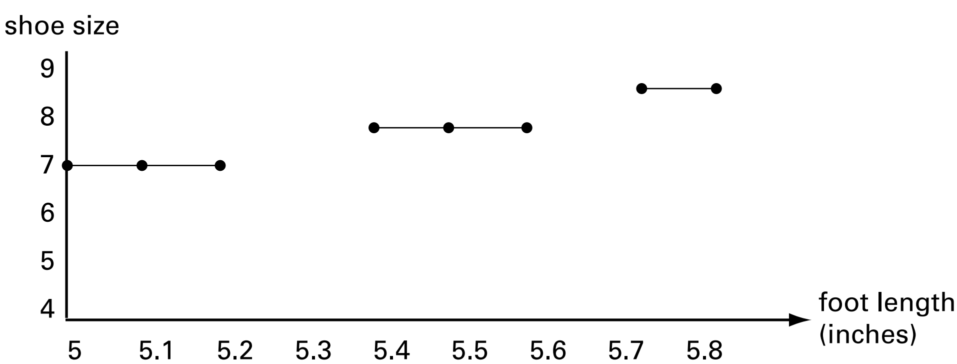
Topic 1 – Functions

Some functions have breaks, for example a graph that shows the relationship between a person's foot length and their shoe size. Foot length is a continuous variable but shoe size is ordered in steps. The function that relates (or maps) foot length to shoe size is a **step function**. If we were to match foot size to mukluks, however, the relationship would be continuous because a mukluk can be made to any foot size.

A table mapping the input variable, foot length, to the output variable, shoe size, might look like this:

Foot length	shoe size
5"	7
5.1"	7
5.2"	7
5.4"	7.5
5.5"	7.5
5.6"	7.5
5.7"	8
5.8"	8

A graph of this function would look like this:



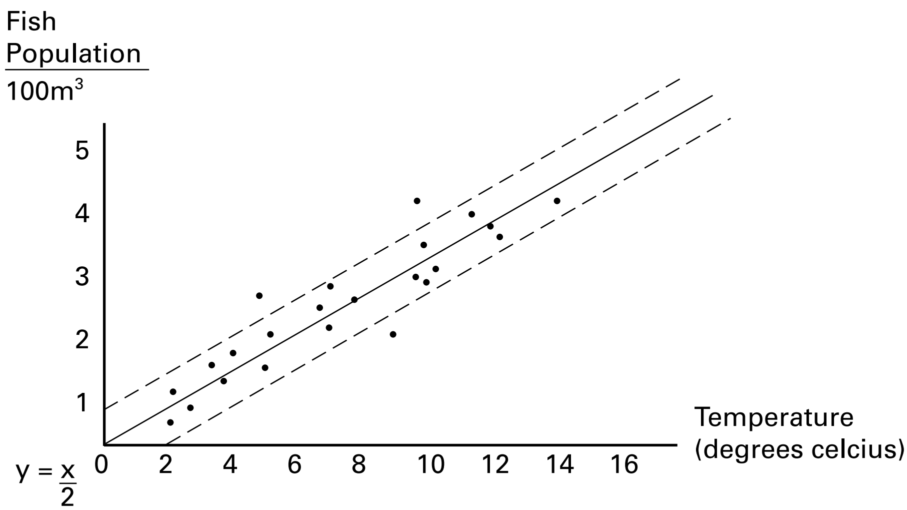
Notice that shoe sizes are not given for 5.35" and 5.65" in the table, yet some feet will have these lengths. A choice would have to be made as to whether the larger or smaller size is chosen as the output number for these inputs. Shoe size is a discrete variable, and foot length is a continuous variable. The graph shows a "step function" because a line cannot be drawn between the y values for some input values, for example between the shoe size that correspond to 5.3" and 5.4". Additional instructions will be needed to know what shoe size to assign to a foot that measures 5.3". A well defined step function dealing with shoe size and foot length will indicate whether to put an input value into the nearest lower or upper shoe size.

Unit 1 – Equations and Patterns

Topic 1 – Functions

Where do functions come from?

Functions can come from answers to questions about nature. For example, a wildlife conservation officer may want to do an environmental impact study to assess how changes to the temperature of a lake caused by a mine will affect fish population. These risks can be assessed by predicting how changes in temperature will cause changes in fish population. The data gathered from this study would produce a **scatter plot**. Not all of the data points will lie on a line, but they may approximate a line that can be described by an equation.



Each point is an observation made during the study from a sampling of northern lakes that were chosen because they were similar in size, fish varieties, latitude, and food supply. With these variables under control, variations in population can be attributed to temperature differences- unless other factors need to be considered as well, for example the acidity (ph) of the lake water.

Good scientific practice will use the equation that describes (ie correlates) the relationship between two variables to make predictions. If the predictions turn out to be true, then the correlation gains support and one variable is proven to be a function of the other one. This is how scientific "laws" are proven and also how they can be disproved.[5]

The dotted lines indicate a band within which the data points fall. Because the lines are parallel, the data describes a **linear relationship**. A straight line graph midway between the upper and lower bounds is one way to approximate **the line of best fit**. The equation $y = \frac{x}{2}$ describes this line. It can be used to predict the likely fish populations of similar lakes when the temperature is known. Formulas are based on observations that reveal functional relationships in scatter plots.[6]

5 Positive correlation should not be confused with causality. The fact that two events are paired, i.e. are usually observed together, does not prove that one causes the other.

6 The ideal gas laws, discussed in Science II, are examples of formulas derived from scatter plots. Physical laws in general are derived from experiments that produce a scattering of data

Unit 1 – Equations and Patterns

Topic 1 – Functions

Functions can approximate real situations

A portion of this graph could approximate the relationship between fish populations and the temperature of Northern lakes. **In applied situations there will be upper and lower limits to the values that a function can take on**. For example, as temperature increases, there will come a point when the fish population declines. However, between a range of temperatures, say 5 degrees Celsius and 20 degrees Celsius, the equation $y = \frac{x}{2}$ will be close to the data given in a table where x measures temperature and y measures fish population. Functional relationships "fit curves" to the data given in tables. Within limits these equations can be used to make predictions.

2. Some tables use the output value of one row as the input value for the next. A function based on this pattern is called a **recursive function**. For example the function $y = \frac{x}{2} \times 3$ would produce the following table if the output number for each row is used as the input number for the next row.

x	f(x) = y
10	15
15	22.5
22.5	33.75

Notice that y = f(x) means **we can write any equation as a function of x.** For example $f(x) = \frac{x}{2} \times 3$ means the same thing as $y = \frac{x}{2} \times 3$.

Here y is a function of x, and each value for y is the result of taking an input for x, dividing it by two, and multiplying by three. A business that is expanding rapidly, for example Nortel in the 1990's, might use monthly sales figures at the end of each month as inputs into a re-supply formula for the following month. The formula might be $y = x + \frac{1}{3}x$, which means the next month's inventory will be increased one third over the previous month.

Unit 1 – Equations and Patterns

Topic 1 – Practice Exam Questions

Question 1

Which ordered pair is a solution for $y = 10x - 2$?

a) (10, -2)

b) (20, 198)

c) (-2, 10)

d) (-20, -100)

Answer: b

Explanation

Choice b tells us that $x = 20$, and $y = 198$. Put these values into the equation and you will get a true statement: $198 = (10)(20) - 2 = 200 - 2 = 198$. Choice b names an ordered pair in the solution set for the function. The other choices do not.

Question 2

A scatter plot shows that concrete hardens more slowly as temperature increases when an additive is used. What functional relationship could you give an approximate equation for?

a) Temperature as a function of additives.

b) Concrete temperature as a function of time.

c) Rate of hardening as a function of temperature.

d) Rate of temperature change as a function of additives.

Answer: c

Explanation

The question describes a relationship between rate of hardening and temperature. As temperature increases it will take longer for the concrete to harden. We label the x axis temperature, and the y axis time for hardening. As x increases so will y. y is a function of x because hardening time depends on the temperature in this example. It is important to see that the reverse is not the case: temperature does not depend on hardening time – also it is correlated to it. The x axis is used to represent the variable that is "responsible" for the y value. Mathematically we say that x is the independent variable, and y is the dependent variable. Y depends on x for its value.

Unit 1 – Equations and Patterns

Topic 1 – Practice Exam Questions

Question 3

A table for $y = x^2$ uses the output from each row as the input for the next row. If the input for row 5 is 10, what will be the input for row 6?

a) 25

b) 66

c) 100

d) 20

Answer: c

Explanation
This is a recursive function. The output, or y value, for row 5 will be 10 x 10 = 100. This is the input for row 6. You can also see that the input for row four is $\sqrt{10}$ because $(\sqrt{10})^2 = 10$, and 10, the output for row 4, is given as the input for row five in the problem.

Question 4

What is the correct value needed to complete this table:

x	y
10	20
5	10
20	40
100	200
3	?

a) 6

b) 15

c) 30

d) 18

Answer: a

Explanation
Each y value equals twice the x value for a row in the table. The rule for this function is $y = 2x$. Every input number is doubled to give its corresponding output value.

Unit 1 – Equations and Patterns

Topic 1 – Practice Exam Questions

Question 5

What is the equation for the function given by the table in the last question?

a) $y = x/2$

b) $y = x^2$

c) $y = 2x$

d) $y = 3x/2$

Answer: c

Explanation
See explanation for question four.

Question 6

Which of the following is a step function?

a) The volume of water in a lake as a function of rainfall.

b) The amount of hair on a rabbit as a function of its size.

c) The amount of income tax as a function of income.

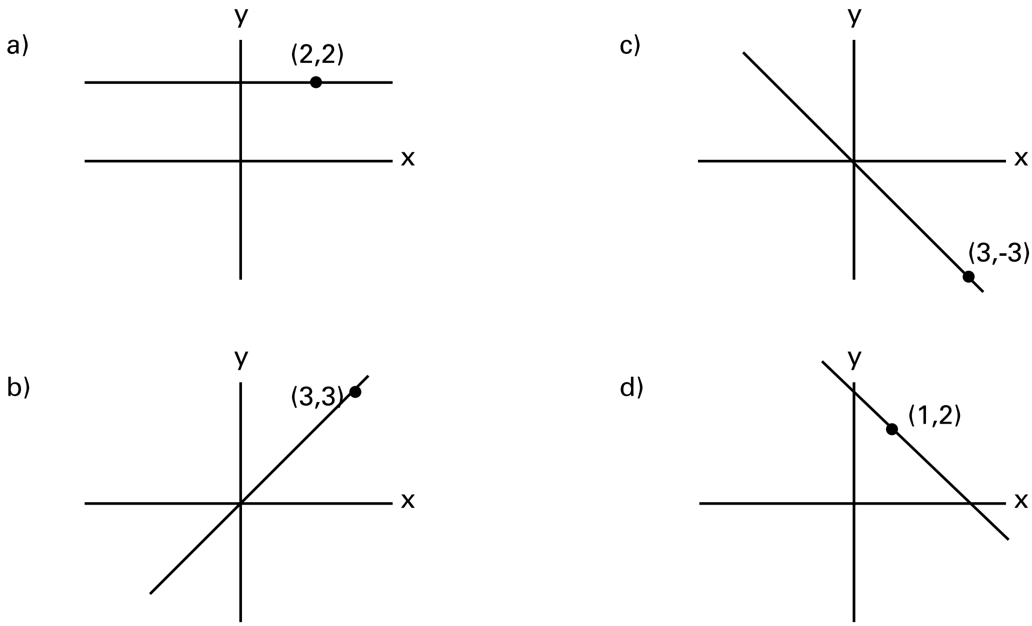
d) The height of a person as a function of their age.

Answer: c

Explanation
A step function is one that does not offer a specific output value for every input. Several inputs may have the same output, and some inputs will have no output until a rule is given that directs the input number to an upper or lower range of values. The example of foot length and shoe size was given earlier in this topic. Income tax is graduated in brackets. Several input incomes will correspond to the same amount of tax. The intervals used for tax are closed, meaning that endpoints are designated for each bracket so that any income will fall into one, and only one, bracket.

Question 7

Which graph shows the function y= x, i.e. f(x) = x



Answer: b

Explanation
Try a few ordered pairs that satisfy the equation. Since y = x, (-1, -1), (0,0), (1,1) are on the graph. This equation is in the form y = mx + b discussed in Module 2 and will produce a straight line graph. Only two points are needed to sketch this graph because it is a line. The graph will make a diagonal 45° above the x axis through the origin.

Topic 2 – Trigonometric Functions

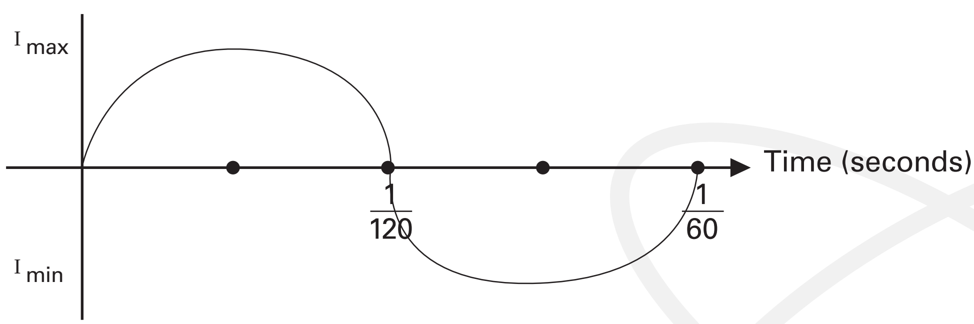
In this topic trigonometric functions will be used to study patterns that repeat themselves at regular intervals.[7] For example, the motion of a point on a wheel (gear, turbine) can be described with the sine function. For this reason they are also known as **cyclic or periodic functions**. Review the material on the trigonometric ratios (sine, cosine, and tangent) covered in Math 4 for this topic. It will be helpful to use a calculator to check the relationships described in this topic.

Periodic Functions

Any process that repeats itself between a range of predictable values is cyclical or periodic. The orbits of the planets, the behavior of electrical current in a circuit, and the motion of a weight on a spring are examples of periodic functions.

> **Periodic Functions**
>
> "What goes up must come down."
>
> "What goes around, comes around."
>
> Periodic functions are cyclical: they repeat the same pattern over and over again.
>
> y = sin x and y = cos x are periodic functions.

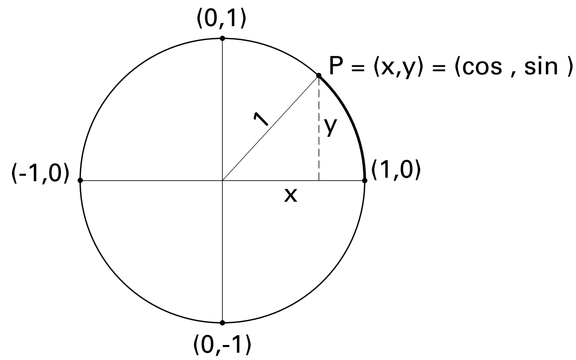
For example, alternating current reverses direction many times each second. 60 cycle AC current changes direction (i.e. polarity) 60 times each second. This means that the amount of current fluctuates between a maximum amount (measured in amps) and a minimum amount 60 times each second. The same pattern is completed every sixtieth of a second. This pattern can be represented by a sine curve based on the sine function. A table of x values for time and y values for current at each choice of x will produce a sine curve. The x axis will represent time, and the y axis will represent the amount (amplitude) of the current at each instant.



7 The trigonometric functions expand the idea of the trigonometric ratios covered in Math 4 – Unit 2, Topic 3.C – Ratios in Right Triangles: Trigonometry.

Topic 2 – Trigonometric Functions

This sine wave will complete 60 cycles each second. Its **frequency** is 60 cycles per second. The current produced by a conductor (i.e. a loop of wire) cutting lines of magnetic force in a generator will also fit a sine curve. The following discussion will explain how to interpret sine waves and other periodic functions. We begin with a review of the unit circle and the correspondence between points on its circumference and points on the graph of a periodic function.[8]



The Unit Circle

The terminal side of a central angle will intersect a point P on the circumference of a circle with a radius of one that has a horizontal x coordinate, and a vertical y coordinate. You can see that P = (x,y). The signs of x and y will depend on which quadrant P is in. This point is known as the trigonometric point. In a circle with radius = 1, by definition, the x coordinate is the cosine and the y coordinate is the sine of the central angle q corresponding to P. Cos q = x/1, and sin q = y/1. P is also the terminal point of the arc that is intercepted by a central angle. Circles with a radius other than one will have coordinates for P that are proportional to the values of x and y on the unit circle.

This important fact is captured in two formulas that should be memorized:

x = r cosq

y = r sinq

In the case of the unit circle r =1, and x = cosq, y = sinq

For Further Clarity:

The central angle will remain the same as the radius and corresponding arc length and circumference change. Study the next diagram to see that the coordinates for any point P on the circumference of a circle of any size are

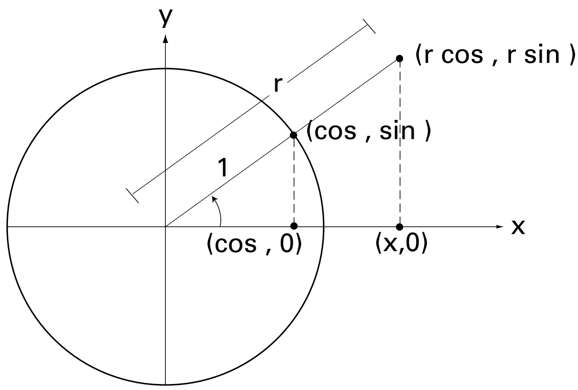
x = r cos q

and

y = r sin q

8 A periodic function takes the form f(x) = f(x + a) where a is a nonzero constant. The smallest positive value for a gives the period of the function. The function will produce the same output value for x + integer multiples of the smallest value for a.

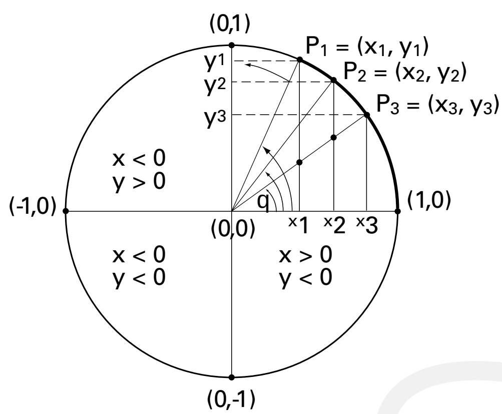
Topic 2 – Trigonometric Functions



The two dotted perpendicular lines lay out two similar triangles with the same angles. Therefore, by similarity,

$\frac{x}{r} = \frac{cosq}{1}$ and x = r cosq, and $\frac{y}{r} = \frac{sinq}{1}$ and y = r sinq.

Study the next diagram to see two things: first how the relationship between the size of x and y change as the central angle increases and completes one revolution, and second how the trigonometric ratios for q on the larger circle reduce to the same number on the unit circle because q doesn't change as the radius changes and the sides of similar triangles are proportional.[9]



The trigonometric ratios reduce to the x and y coordinates of the unit circle because the heights (y values) correspond, and the radii correspond.

$$sinq = y_1 = \frac{y_2}{r} = \frac{y_n}{r_n}$$

$$cosq = x_1 = \frac{x_2}{r} = \frac{x_n}{r_n}$$

$$tanq = \frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_n}{x_n}$$

9 Review the topic on similar triangles in Math – Module 4.

Topic 2 – Trigonometric Functions

The unit circle is the basis for all trigonometric ratios, provided that we do not divide by zero. Notice that x cannot be zero in the tangent for q. The y axis is also the line x = 0, and this axis is an asymptote for the tangent: as q approaches 90° tan q approaches, but never reaches, x = 0. See the graph on page 42 for details.

As the radius rotates in the diagram, the coordinates for P_1 on the unit circle will change continuously in a range of values for x and y from –1 to +1. In the first quadrant x and y are both positive and range between 0 and +1. In the second quadrant, x is negative and y is positive. In the third quadrant x and y are both negative. In the fourth quadrant, x is positive and y is negative. The sign of the sine, cosine, and tangent of the number based on the size of the central angle will vary according to which quadrant the x and y coordinates of the point P are in.

The resulting effect on the sign of the trigonometric ratios can be remembered by an acronym "**A**ll **S**tudents **T**ake **C**ourses, meaning that all functions (sine, cosine, tangent) are positive in the first quadrant, only the sine is positive in the second, only the tangent positive in the third and only the cosine positive in the fourth.

Angles Have Direction

Angles are measured from the positive x axis. The sign of an angle tells us if we move the radius counterclockwise (+) or clockwise (–) from the positive x axis. P= (1,0) is where the point starts to move from to create an angle on the unit circle. 30° has a terminal side in the first quadrant, 120° has its terminal side in the second quadrant, and –120° has its terminal side in the third quadrant.

A More Detailed Description: The Coordinates of P Change As The Radius Rotates

If you imagine the radius rotating counterclockwise on a unit circle, you can picture how the central angle, theta, (q) will increase and the lengths of x and y will change. As the radius rotates, the x coordinate of the point P will decrease and start to become negative as the point goes past the 90 degree mark into the second quadrant. Meanwhile, the y coordinate will increase to a maximum of 1 at the ninety degree mark when y = r = 1. As P enters the second quadrant, y becomes smaller than one, but remains a positive number. As the radius rotates it moves P through every point on the circumference.

Sine And Cosine Are Always Between –1 and 1

The trigonometric functions map (ie. have a rule of correspondence) between any angle and the ratio of two of the sides of the right triangle that can be formed with the angle on a unit circle. Sine, cosine and tangent functions can be studied by picturing a rotating radius in a unit circle. As the radius = 1 travels counter clockwise from the point (1,0), the side opposite the angle made at the origin increases until 90 degrees is reached. At this point, one fourth of the circumference has been covered (i.e. $\frac{2\neq r}{4} = \frac{\neq r}{2} = \frac{\neq}{2}$)

and the sine of 90° = 1, the cosine of 90° = 0, and the tangent of 90° is undefined because it would involve dividing by 0.

Topic 2 – Trigonometric Functions

Positive and negative integers, fractions, and irrational numbers can be input numbers, i.e. can be values of q, for the trigonometric functions. The domain of the trigonometric functions includes all the numbers studied in this curriculum, but the range of the sine and cosine are between –1 and +1 because all the ratios reduce to the ratios found on the unit circle.

We Use Six Trigonometric Functions

Three related functions are defined in terms of the primary functions: the cosecant is related to the sine, the secant to the cosine, and the cotangent to the tangent. On a calculator 1 ÷ sineq = cscq, 1 ÷ cosineq = secq, and 1 ÷ tangentq = cotq.

Important: Reciprocal Functions Are Not Inverse Functions

On a calculator do not use $\cos^{-1}$ q, $\sin^{-1}$ q, $\tan^{-1}$q, $\sin^{-1}$q to find secant, cosecant and cotangent. These are the inverse trigonometric functions aka arcsin, arcos, and arctan, not the reciprocals of sine, cosine, and tangent.

The inverse trigonometric functions are used to find q when the value of sin q, cos q, or tan q is known and can be used as inputs. By the definition of inverse function, the domain of a function becomes the range of the inverse function, and the range becomes the domain. Use a calculator to see that $\sin^{-1}$ q is not equal to 1 ÷ sin q.[10]

The three related functions are not defined for division by zero, and input values that would cause this to happen are excluded from their domains as noted in the following table.

For k = any integer.

secant: secq = $\frac{1}{\quad}$ x ↑ $\frac{≠}{2}$ + k≠, examples: q ↑ 90°, 270°, 450°… etc.

cosecant: cscq = $\frac{1}{\quad}$ x ↑ k≠, examples: q ↑ 0°, 180°, 360°, 540°… etc.

cotangent: cotq = $\frac{1}{\quad}$ x ↑ k≠, examples: q ↑ 0°, 180°, 360°, 540°… etc.

Only the sine and cosine allow any real number (–• , •) to be an input value, i.e. to be included in the domain of the function. Division by zero is undefined, and this means that some values are not included in the domains of the functions that involve division by the sine or cosine.

10 You will use the inverse functions to solve some trigonometric equations later in this topic, but we do not study their graphs.

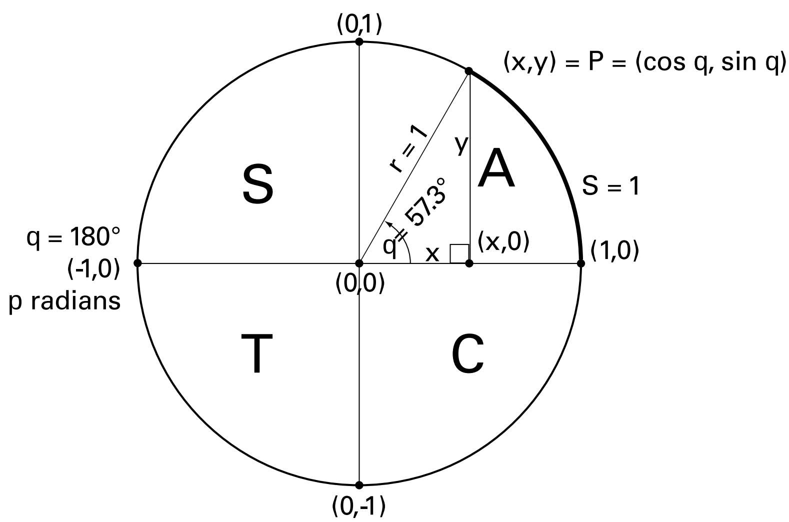
Topic 2 – Trigonometric Functions

Notice that the domains of the functions related to the sine and cosine (i.e. tangent, cotangent, secant, and cosecant) are not defined for all real numbers. The tangent and secant could involve division by the cosine, and therefore have undefined meaning for q = 90° and all additional multiples of 180° where the cosine = 0. Similarly, the cotangent and cosecant involve division by the sine, and are undefined when sine = 0 at 180° and its multiples.

To understand this better, keep the unit circle in mind and notice that sinq = y = 0 on the x axis (at 0°, 180°, 360°), and cos q = x = 0 (at 90°, 270°), on the y axis. The secant and cosecant have no output values (their range) between -1 and 1. These facts will be illustrated in the graphs of each function given later.

Trigonometry Combines Geometry With Algebra

The next diagram contains a lot of information that you will need. It shows the connection between a geometrical interpretation of the trigonometric functions and an algebraic interpretation of them in an x, y coordinate system.



Topic 2 – Trigonometric Functions

Recall that **the coordinate system has four quadrants.** The first quadrant includes the central angles from 0 to 90 degrees, the second from 90° to 180°, the third from 180° to 270°, and the fourth from 270° to 360°. In quadrant one, x and y are both positive. In quadrant three, x and y are both negative. In quadrant two x is negative and y is positive, and in quadrant four x is positive and y is negative. Since the x coordinate is the cosine of the corresponding central angle, and the y coordinate is the sine, the signs of the trigonometric ratios in each quadrant can be summarized in an acronym.

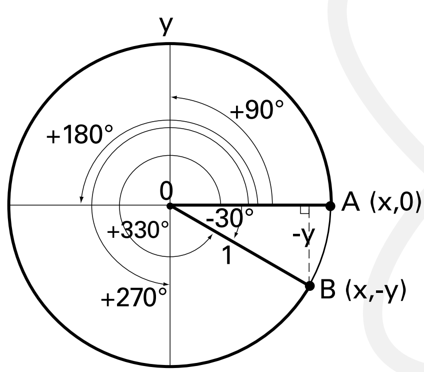
All **S**tudents **T**ake **C**ourses

ASTC will help you remember the sign of each trigonometric function in each quadrant. All three functions (sine, cosine, tangent) are positive in quadrant one, only the sine **(S)** is positive in two, only the tangent **(T)** is positive in three, and only the cosine **(C)** is positive in four. As explained below, you will find the reference angle in the unit circle corresponding to any angle, or indeed to any real number, and put the correct sign for any output of a trigonometric function. For example, the sine of 130° is positive, but the cosine and tangent are negative (quadrant 2), the sine of 210° is negative as well as the cosine but the tangent is positive (third quadrant). Use a scientific calculator to verify this.

Arc Length and Radian Measure

Refer to the diagram above to see that theta (q), refers to the angle made by a radius with the center of a circle and the positive x axis. We will call this the **central angle**.[11] There is only one angle, i.e. only one value for q, that intercepts the part of the circumference (an **arc length**, s) that is equal to the radius of a circle. This angle is 57.3 degrees in a circle of any size. This angle is equal to one **radian**. Identify s, r, and q in the diagram on page 22 and observe the equality between r and s.

In trigonometry think of the radius as a directed line, or **vector**, that can rotate in either the positive counterclockwise direction, or the negative clockwise direction.[12] Each time any radius rotates through 360 degrees it goes around the circumference once. The circumference = 2≠r and the radius will complete one revolution, or **cycle**, and return to where it started from. The **terminal side** of a central angle is the ray that is free to rotate. The central angle made by a radius vector will have both magnitude (the size of the angle) and direction (whether the terminal side rotates clockwise (-) or counter-clockwise (+)).



[11] Some textbooks use a (alpha) or another symbol to represent the central angle.

[12] See Unit 2 – Topic 1.

Here OA is the **initial side** and OB is the **terminal side** of angle AOB. When the unit circle is placed in an x, y coordinate system, the measure of an angle in standard position has the initial side on the positive x axis. Degree measure will be positive when the radius rotates counterclockwise, and negative when it rotates clockwise.

A negative angle means that the radius is rotated in a clockwise direction. A directed radius is also known as a r**adius vector**.[13] The sign of an angle depends on which way the radius rotates. To indicate that the direction of the radius vector is anticlockwise, the angle is positive. Angle AOB measures 330° degrees. Angle –AOB is the angle formed when the radius vector rotates clockwise from OA to OB. Angle –AOB measures –30°. AB is the arc on the circumference that is intercepted by angle AOB.

Co-Terminal Angles and Reference Angles

This diagram shows that an angle of –30 degrees has the same terminal side as does an angle of 330 degrees. **This means they will share the same terminal point on the circumference of the circle**. The terminal side of both angles is in the fourth quadrant. These angles are **co-terminal**. The terminal side of a central angle will always be in one of the four quadrants and end in a point P = (x,y) where the signs of x and y depend on the quadrant P is in.

There are an Infinite Number of Co-Terminal Angles

Any angle larger than 360° will produce a terminal angle somewhere on the unit circle after all complete revolutions (i.e. multiples of 360°) have been counted. An angle of 720° means the radius vector has gone around the unit circle twice to produce the co-terminal angle of 0° on the x axis.

It is also true that any central angle, and more generally any angle formed by two rays, can have infinitely many measures based on the co-terminal angles that cause the radius vector to wind up at the same place.

The **reference angle** related to any angle is found by finding the positive acute angle between the terminal side and the positive or negative x axis. When this angle is used as the input number to find the value of a trigonometric function, the sign appropriate for the quadrant can be added. The trigonometric function of any angle can be found by using its reference angle as an input.

It is a good idea to use a calculator (or your knowledge of exact trigonometric values) to find the value for the reference angle first, and then add the sign that is needed based on the value the function has in the quadrant that is required.

Study the following examples to get clear on the difference between co-terminal angles and the reference angle for a given angle.

[7] See Math – Unit 2 – Topics 1 and 2 on vectors, below.

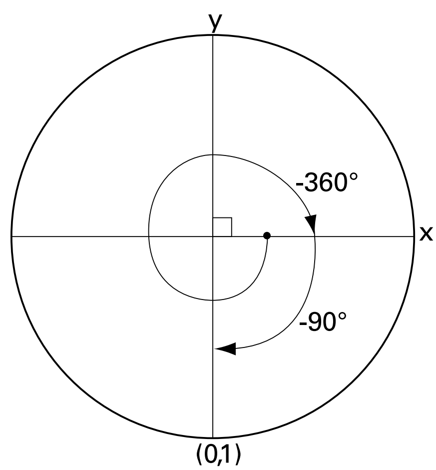
Topic 2 – Trigonometric Functions

Examples

1. 330° is in the fourth quadrant. The reference angle for 330° is 30° because 360 – 330 = 30° and this is an acute angle. The absolute value of the sine, cosine, tangent etc. will be the same for 330° and 30°, but the sign will depend on the quadrant and function. For example, Sin 30° = .5, but sin 330° = -.5. Tan 330° = -.577, tan 30° = .577. 330° is co-terminal with -30°.

2. The reference angle for 720° is 0°, and the trigonometric functions will give the same outputs for both angles. Verify this on a calculator.

3. Use the following diagram to understand that -450 degrees means to go once around **clockwise** to use up the first 360 degrees contained in -450°. The remainder, 450 – 360, is 90 degrees continuing clockwise. This remainder produces a terminal side at -90°, in the fourth quadrant. The coordinates for the point on the circumference of the unit circle at -90° are x = 0, y = -1. the sine of -90° = -1, and the cosine of -90° = 0.



4. -90° and -450° and 270° are co-terminal because they have degree measures (i.e. magnitude of the angle made by the radius vector) that produce the same terminal side on the unit circle. Satisfy yourself that this is so by going around the unit circle the indicated number of degrees in the correct direction on a sketch of the unit circle.

   The reference angle for -90°, -450° and 270° is 90°. These angles are in the third quadrant and correspond to the trigonometric point (0, -1) The cosine will be zero, the sine will be negative, and the tangent will be undefined for all three angles. Check this on your calculator.

   Notice that 270° is co-terminal with -90° and -450°, but that it rotates the radius in the opposite direction. Notice also that 450° is not co terminal with -450° because 450° has its terminal side in the first quadrant and sin 450° = 1, not –1.

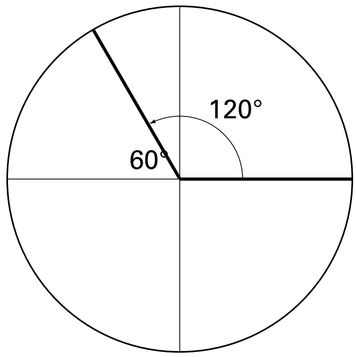
Unit 1 – Equations and Patterns

Topic 2 – Trigonometric Functions

Sample Problems

1. Find the reference angle for 120 degrees.

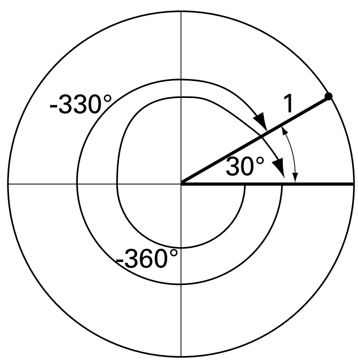
   Sketch the angle on a unit circle:



   The negative x axis is the terminal side of an angle of 180°. Since 120 degrees is positive, the radius rotates counterclockwise and stops 60° before reaching 180 degrees. The reference angle is 180° – 120° = 60°.

2. Find the reference angle for -690°.

   Sketch the angle on a unit circle:



   An angle of -690° goes once around the circle clockwise and then -330 degrees to complete the total of -690°. -330° is in the first quadrant and stops 30° short of the x axis. The reference angle = 360° – 330° = 30°. The next part of this topic will show how to use geometry to find the exact sine and cosine of 30°.

3. Is –50° co-terminal with 670°?

   First locate -50° by rotating the radius clockwise 50 degrees. The terminal side is in the fourth quadrant. Next, find the measure of the positive angle that is co-terminal with -50°. This angle is 360° – 50° = 310°. We need to know if a radius that rotates counterclockwise 670° winds up at the same place as does an angle of 310° aka -50°. You can picture the radius of a 670° angle completing the first 360° revolution and then starting its second trip with 670° – 360° still to go. This difference is 310°, which does make it co-terminal with 310° and -50°.

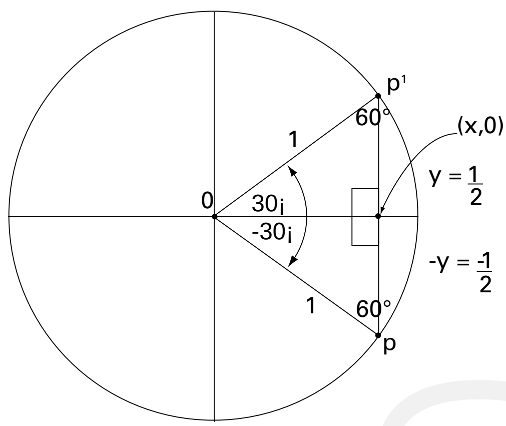
4. What quadrant is 1380 degrees in? What acute angle is it co-terminal with?

   This angle is larger than 360, so we need to know how many degrees remain after all complete revolutions are counted. 1380/360 = 3.83... We can conclude that 83% of a revolution remains after three trips have been completed in the counterclockwise (+) direction. 3 x 360 = 1080, so 1380 – 1080 = 300 degrees remaining. 300° has its terminal side in the fourth quadrant. It is co-terminal with -60 degrees.

5. What is the reference angle when a radius goes through 1100 degrees? Divide 1100/360 to get 3.0555. This means the radius completed three revolutions and .0555 of a fourth revolution. 3 revolutions totals 3 x 360 = 1080°. The remainder available for part of a fourth revolution = 1100° – 1080° = 20°.

The Coordinates of Any Point on the Unit Circle Give Us the Sine and Cosine of the Corresponding Central Angle

In the following diagram, the coordinates of point P equal the values of the cosine and sine of –30 degrees as well as all of its co-terminal angles. Satisfy yourself of this by using the definitions for sine and cosine, i.e. calculate opposite over hypotenuse, and adjacent over hypotenuse. Notice that the cosine will be positive in the fourth quadrant, and sine and tangent will be negative. The following discussion will give you the trigonometric values for key reference angles that should be memorized or recoverable from a sketch of the unit circle: 30°, 45°, 60°, 90°, 180°.



In this example q = 30°. Because the radius of the unit circle is 1, and the hypotenuse of the right triangle containing the central angle is equal to the radius, $\sin q = \frac{\text{opposite}}{1}$ = opposite = y, and $\cos q = \frac{\text{adjacent}}{1}$ = adjacent = x.

The tangent of q $\frac{\sin q}{\cos q} = \frac{y}{x}$ equals by the definition of tangent, ie. $\frac{\text{opposite}}{\text{adjacent}}$.

The Sine and Cosine of 30°, 60° and 45°

Study the diagram above showing a line perpendicular to the x axis connecting P and P′ to see how we can create equilateral triangle POP′ and use the Pythagorean theorem to find the exact values for the lengths of y and x. These values will also give the coordinates for P and P′.

From geometry we know that the angle with vertex P′ and the angle with vertex P are both 60° because the sum of the angles in a right triangle must be 180° and we have central angles of 30 degrees in magnitude.[14] We also know that y = 1/2 and -y = -1/2 because PP′ is the same length as the radius, ie. 1, and the line from (0,0) to (x,0) bisects PP′. Now that we have the length of x, we can use this information to find the sine of 30°= .5, and the cosine of 60° = .5 by the definition of sine and cosine. The sine of -30° is equal to the magnitude of the y coordinate, and turns out to be equal to -.5 in the fourth quadrant. The cosine of -60° is +.5 in the fourth quadrant. (sketch the triangle for this angle to see this).

The Pythagorean theorem leads to the following value for x:

$$x^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

$$x = \sqrt{1 - \frac{1}{4}}$$

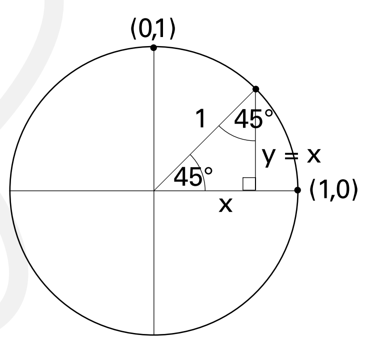
$$x = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

With this information we can see that P′= $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, and P= $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

The tangents (the ratio expressing rise (y) over run (x) of 30° and 60° are found by dividing cosine into sine, which equals the opposite over the adjacent sides of the right triangles formed by the central angles involved.

$$\tan 60° = \frac{\sin 60°}{\cos 60°}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \sqrt{3} \quad \text{and} \quad \tan 30° = \frac{\sin 30°}{\cos 30°}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \frac{1}{\sqrt{3}}$$

A similar analysis will prove that the coordinates of P for 45° = $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



[14] See Math 4 – Measuring Time, Shapes, and Spaces:Triangles.

The two sides of the right triangle are equal in a 45°, 45°, 90° triangle. Therefore x = y and the Pythagorean relationship proves that:

$x^2 + x^2 = 1$

$2x^2 = 1$

$x = y = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Notice that the second form of the answer "rationalizes the denominator" by multiplying by a form of one:

$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

This is a common practice in textbooks and on exams - but both answers are correct, and different problems will make one form the most convenient one.

What You Need to Know: Exact values

	$30° = \frac{\neq}{6}$	$45° = \frac{\neq}{4}$	$60° = \frac{\neq}{3}$	$90° = \frac{\neq}{2}$
sin	$\frac{1}{2}$	$\frac{1}{\wp 2} = \frac{\wp\overline{2}}{2}$	$\frac{\wp\overline{3}}{2}$	1
cos	$\frac{\wp\overline{3}}{2}$	$\frac{1}{\wp 2} = \frac{\wp\overline{2}}{2}$	$\frac{1}{2}$	0
tan	$\frac{1}{\wp 3} = \frac{\wp\overline{3}}{3}$	1	$\wp\overline{3}$	——

Practice Problems

1. Find the value of all six trigonometric functions for q = 90°.

   Begin with sine and cosine. Locate 90° on the unit circle. You will see that the coordinates for the point corresponding to 90° are (0,1). Sin 90° = 1, and cosine 90° = 0. The third primary function, the tangent, is undefined at q = 90° because tangent = sine/cosine, and division by zero is undefined.

   90° is excluded from the domain of the tangent function. If you ask a calculator for tan 90° it will return "math error" or a similar message.

   The cotangent is related to the tangent and equals $\frac{1}{\tan q}$.

   You might think the cotangent is undefined as well, but notice that

   $\frac{1}{\tan} = \frac{1}{\frac{\sin}{\cos}} = \frac{\cos}{\sin}$. This involves division into zero, not by zero, for q = 90°. Cot 90° = 0.

The cosecant = 1 because $\frac{1}{\sin 90°} = 1$.

The secant is undefined because $\sec q = \frac{1}{\cos q}$ and cos q = 0.

2. What is the secant of 45°?

We know that the coordinates for P corresponding to q = 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

The secant of 45 degrees $= 1 \div \cos 45° = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} \times \frac{2}{\sqrt{2}} = \sqrt{2} = 1.414$

Check this on your calculator.

3. Find the six trigonometric values for a central angle of 210°.

Step One: This angle is in the third quadrant where only the tangent is positive

StepTwo: The reference angle is 210° – 180° = 30°

Step Three: Find each trigonometric function for 30° and attach the correct sign

sin 210° = -sin 30° = -1/2

$\cos 210° = -\cos 30° = -\frac{\sqrt{3}}{2}$

$\tan 210° = \tan 30° = \frac{\sqrt{3}}{3}$

csc 210° = 1/-sin 30 = -2

$\sec 210° = 1/-\cos 30 = -\frac{2}{\sqrt{3}}$

$\cot 210° = \cos 30/\sin 30 = \sqrt{3}$

Verify these on your calculator.[15]

Arc Length

A central angle will intercept an arc that is part of the circumference of a circle. Since the circumference of a circle = 2≠r, and r = 1 on the unit circle, a complete trip around a unit circle will be 2≠, (or 360 degrees) and any central angle q less than an integer multiple of 2≠ will intercept part of the circumference equal to $\frac{q}{2\neq}$. For example, a central angle of 45 degrees intercepts one eighth of the circumference of a circle because $\frac{360}{45} = 8$ and 8 x 45 = 360. One eighth of 2≠r is $\frac{\neq}{4}$ r. On the unit circle this will equal $\frac{\neq}{4}$. In general the arc length, s, intercepted by a central angle is the product of the circumference and the fraction of the circumference on the unit circle intercepted by the central angle.

$s = \frac{q}{2\neq} \times 2\neq r = rq$, therefore

Arc Length = radius x central angle in radians

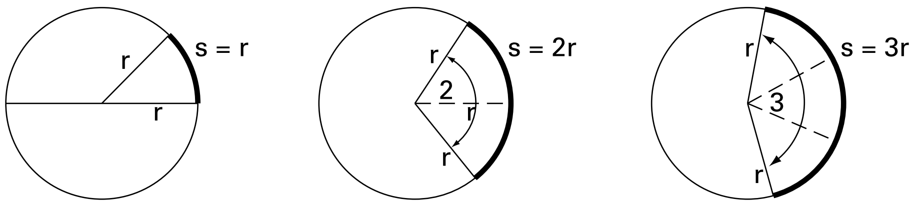
$$s = rq$$

This equation should be memorized.

15 The point (≠/2,0) is on the graph of cot q. However, a calculator may return "math error" for 1 ÷ tan 90° because tan 90° is undefined. When you calculate cot 90° by dividing sin 90° into cos 90° you will get the correct value for cot 90° = 0.

Topic 2 – Trigonometric Functions

The Radian Measure of Angles



You can see that there is a one to one relationship, i.e. direct proportion, between arc length and the size of the central angle. When the arc length doubles so does the angle and vice versa. If we let the central angle, q, equal one when r = s then it follows that

$= \frac{r}{r} = 1$

$2 = \frac{2r}{r} = 2$

$3 = \frac{3r}{r} = 3$

$n = \frac{nr}{r} = n$

This means that the number of radians = arc length divided by the radius of any circle.

The term radian is close in meaning to "radius" because it compares a piece of the circumference (a curved line) to the radius (a straight line). When the radius and the arc intercepted by a central angle are equal, the arc length is 1 radian (corresponding to one radius). When this happens an angle of 57.3 degrees is involved. **An angle of 57.3° is one radian wide. All other arc lengths will be a multiple or fraction of one radian.**

When the circumference is divided into sections of length equal to one radian, there are 2≠ (about 6.28) sections. This relationship restates the fact that C = 2≠r . This means there are ≠ radians in a half circle and we can connect a real number to any angle. **Every angle can be described in terms of the arc length that it intercepts.**

A Radian Has No Dimension, It Is a Scalar Quantity

An **arc length** is part of the circumference of a circle.

A **radian** is a measurement of the angle that corresponds to arc length. In a unit circle r =1, and then arc length (s) = q. A complete circle has an arc length of 2≠ radians or 360°.

The number of radians corresponding to a central angle will equal arc length divided by the radius.

$q = \frac{s}{r}$

Topic 2 – Trigonometric Functions

Degrees of an angle can be mapped onto lengths and vice versa by comparing arc length to radius. Notice that a radian is not a measurement unit like inches or meters. Because there are 360 degrees swept out by a radius to complete a circle, the arc length intercepted by 1 degree equals 2≠ / 360 = ≠ / 180 = .01745 rounded to the nearest hundred thousandth. **The arc length of one degree = .01745… Any angle can be measured in degrees or radians.** For example, an angle of 20 degrees = 20 x .01774 = .3548 radians, or about a third of one radian. One radian is equal to a central angle of 57.3 degrees. Use a calculator to verify that 2≠ x 57.3 equals 360°.

Converting Angles and Radians

> **Optional: How To Find A Radian**
>
> Measure the radius of a circle. Use a piece of string equal to this length and place it on the circumference of the circle. Draw a line from each end of the string to the center of the circle. With a protractor measure the angle at the center. You will find that this angle = 57.3 degrees = 1 radian. **One radian (57.3 degrees) is the measure of an angle that corresponds to an arc length equal to the radius of any circle.**
>
> Think of a radian as the angle size that produces an arc length equal to the radius of a circle. You will get the same result no matter how big the radius is. This relationship allows us to connect any angle to the arc length it subtends and to express any angle in terms of radians and vice versa.
>
> This is useful because any repeating process can be mapped onto a unit circle. Each cycle of the process will correspond to one trip around the circumference, or 2≠ radians of arc length.

Angles can be measured in degrees or radians. You have seen above, that if s is the arc length corresponding to the angle q, r is the length of the radius, and q is the size of the central angle in radians, then: s = rq

This says that the arc length corresponding to a central angle in a circle is equal to the radius times the size of the central angle.

The proportion showing the relationship between radians and degrees is

$\frac{q}{180} = \frac{\text{radians}}{\neq}$ where theta is in degrees.

Unit 1 – Equations and Patterns

UNIT 1

Topic 2 – Trigonometric Functions

Unless degrees are specified, the trigonometric functions are expressed in radians.

$$1 \text{ radian} = \frac{180°}{\neq} \text{ and } 1° = \frac{\neq}{180°}$$

Sketch What You Know

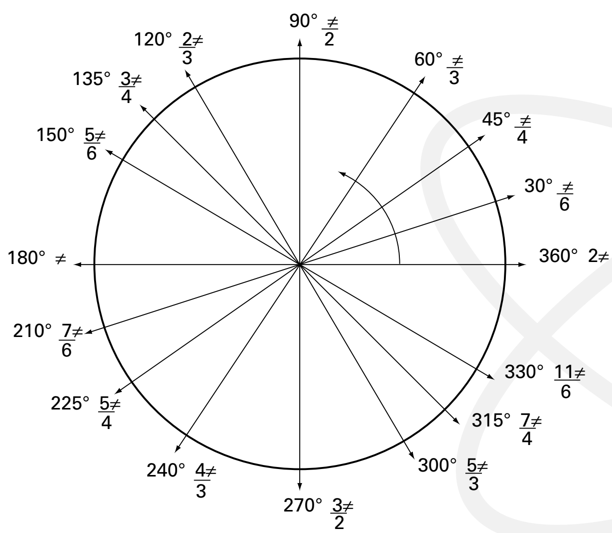
You can always recover your understanding of the relationship between radians and degrees by sketching a unit circle and remembering that C = ≠D. D = 2 in a "unit circle" because the radius is 1. Once you have C = 2≠, you can find what any part or multiple of C equals by performing the same operation on both sides of the equation.

For example, if you want to find the number of radians in two revolutions around the unit circle, multiply 2≠ by 2 to get 4≠ radians or 720 degrees. If you want to know how many degrees are equal to ≠/6 radians, use the proportion

$$\frac{q}{180} = \frac{\text{radians}}{\neq}$$ taking theta in degrees as the unknown. Then

$$q = \frac{180r}{\neq} = \frac{180\neq}{6\neq} = 30 \text{ degrees.}$$

Important relationships between angles and arc length (portions of the circumference) that you need to know are summarized in the next diagram. These relationships provide input values that are important for graphing trigonometric functions. If you memorize the angle and arc length points in the first quadrant you can construct all of the other points because they are multiples of those in the first quadrant.



Unit 1 – Equations and Patterns

Topic 2 – Trigonometric Functions

Study this diagram to recognize the symmetries involved. Each ray will go through two points on the unit circle's circumference. For example, the coordinates (i.e. the cosine and sine) for P that correspond to 60° and 240° are the same except for sign. Verify these relationships on your calculator.

Examples:

1. Find the radian measure of 30 degrees.

   30 degrees is 30/360 or 1/12 of the 360 degree angle corresponding to the circumference of a circle. 1/12 of the circumference of a unit circle = 1/12 x 2≠ = ≠ / 6 radians. In this way the size of any angle is mapped onto a corresponding length.

2. Find the number of degrees in 10 radians and calculate the quadrant of the terminal point and the reference angle.

   Use $\frac{q}{180} = \frac{\text{radians}}{\neq}$ and solve for q

   $$q = \frac{180 \times 10}{\neq} = \frac{1800}{\neq} = 573°$$

   You can also solve this problem using the conversion factor 1 radian = 57.3°.

   10 x 57.3 = 573°. This angle is equal to one trip around a unit circle plus

   573° – 360° = 213° . To find the reference angle, sketch 213° and find the difference between 213° – 180° = 33°. 213° is in the third quadrant where x = cos 33° will be negative, and y = sin 33° will be negative, y/x = tan 33° will be positive. 213° and 33° are co-terminal and will have the same absolute trigonometric values.

3. What is the angle formed when ≠ radians is added to 120 degrees? What is the sine and cosine of this angle?

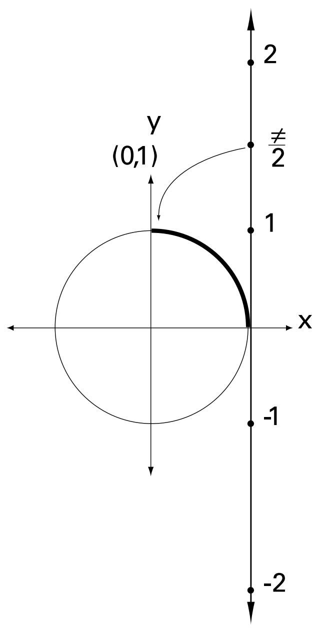
   We need an answer in degrees, therefore change ≠ radians into degrees and add. ≠ radians = 180°. 180° + 120° = 300°. 300 degrees is in the fourth quadrant where sin x will be negative, and cos x positive. **In general, note the sign of an answer before calculating the sine and cosine.**

   Now we need to find the reference angle for 300°. The angle is positive and will have a terminal side 60° below the positive x axis, i.e. 300° is co-terminal with -60°. The sine of the reference angle 60° = $\frac{\sqrt{3}}{2}$, and the cosine of 60° = $\frac{1}{2}$. The final answer attaches the correct signs for the fourth quadrant as required for 300°, sin -60° = - $\frac{\sqrt{3}}{2}$, and cos -60° =1/2. These numbers are also the y and x coordinates, respectively, of the trigonometric point on the unit circle that corresponds to a central angle of 300 degrees.

If you forget the sine and cosine for an angle, you can reconstruct the right triangle on the unit circle corresponding to it and use the methods given earlier to solve for the sides. Alternatively, you can use a calculator or a table and give the answer in decimal form.

Another Way to Think: Sine and Cosine Wrap the Real Numbers Around the Unit Circle

The real number line can wrap around a circle. Every 2≠ the same point on the unit circle is reached. Many real numbers map onto the same point on the unit circle. When a vertical line representing the number line is placed tangent to the circle with 0 at the point (1,0) on the circle, you can imagine how there will be a unique point on the circumference corresponding to every real number on the number line when the line is "wrapped" around the circle. The trigonometric functions can be understood as performing a "wrapping function".



In this diagram you can see how ≠/2 on the number line wraps onto (0,1). The arc length of one fourth of the circle equals ≠/2 on the number line.

You can also see that the number line is infinitely long and can be wrapped around the unit circle an infinite number of times. Every number will have a point on the circle that it wraps to. This is why the trigonometric functions are called circular or periodic. **There will be many real numbers (each different but the same distance apart from each other) that map onto the same point on the unit circle.**

For example, the point (-1,0) on the unit circle corresponds to the central angle of 180° or ≠ radians. On the number line, the point ≠ will touch this point on the circle when it is "wrapped" around the circle.

Unit 1 – Equations and Patterns

Topic 2 – Trigonometric Functions

An infinite series of numbers that wrap to the same point can be formed. They will all be integer multiples of ≠. All of the input numbers that wrap to the same point will have the same x and y coordinates, and consequently the same cosine and sine output values. This is why the general solution for trigonometric equations will have an infinite number of answers. For example, ≠, 3≠, 5≠, 7≠ etc… all correspond to 180° or ≠ radians on the unit circle. The sine will be 0 and the cosine -1 for each of these input numbers.

For example, the solution to the equation cos x = -1 will include all x such that x = ≠ or ≠ + k2≠ where k is a positive integer. Test this on a calculator with some multiples of 2≠.

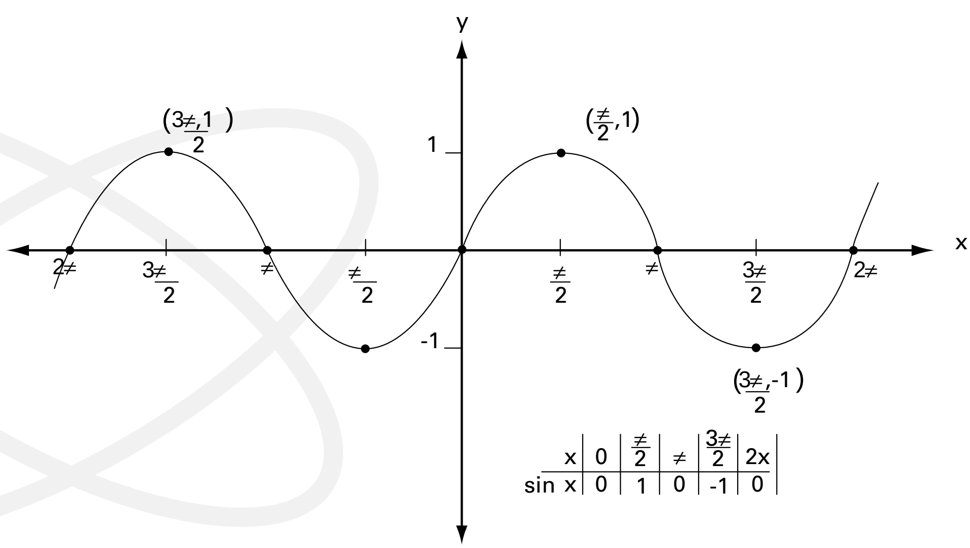
In the language of functions, the domain for the "wrapping function" (i.e. sine and cosine) is all real numbers, and the range is the values between -1 and 1. Picture a moving wheel rolling along the number line to see the periodic nature of the wrapping function. The same point on the wheel's circumference will touch all points on the number line that are integer multiples of 2≠ from its starting point.

Cyclic Patterns: The Sine Wave

The sine curve, or **sine wave,** is a cyclic pattern that can be described using the y coordinate of the trigonometric point on the unit circle as the output for any x chosen for an input. In the example of a moving wheel, x was elapsed time, and y was the distance from a point on the circumference to the horizontal diameter. You can also understand that the sine wave graphs the coordinates of the radius vector.

In this discussion x and q refer to the same variable. To graph the sine function, create a table with input values from the key points around the unit circle that were given above. We are interested in the output numbers over one revolution around the unit circle so we choose values for x from 0 to 2≠. **Radian measure will be used unless degrees are specified.** You should be able to picture the angles associated with the radian measures chosen in the following tables.

y = sin x



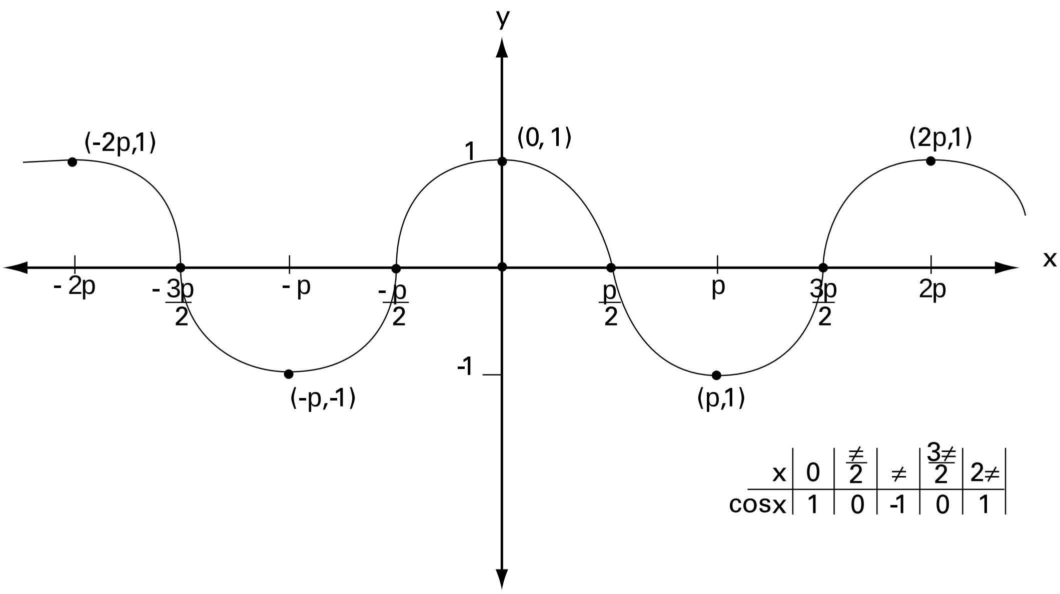
x	0	≠/2	≠	3≠/2	2x
sin x	0	1	0	-1	0

Topic 2 – Trigonometric Functions

Y = sin x is a **periodic function.** You can see that the graph continues indefinitely in both directions. It completes one cycle every 2≠. The smallest interval over which the function completes one cycle is called the period. The **fundamental cycle** of y = sin x is the graph from 0 to 2≠.

Notice that the sign of the values for sin x are reversed on the negative side of the x axis. Use earlier diagrams of the unit circle to verify the values of sin x when x is negative.

Next graph y = cos x and compare with y = sin x.



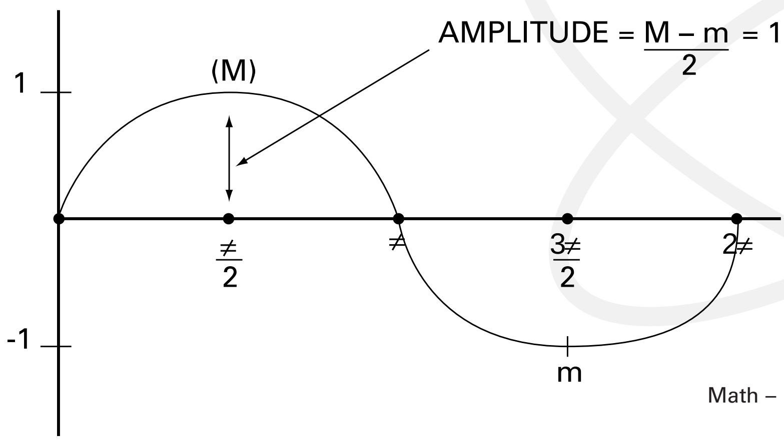
x	0	≠/2	≠	3≠/2	2≠
cosx	1	0	-1	0	1

The graph of y = cos x has the same shape as sin x and the two graphs would coincide if y = sin x is shifted ≠/2 to the left or 3≠/2 to the right. **The cosine is also a sine curve.** Both functions complete their fundamental cycle over 0 to 2≠., which is one revolution around the unit circle.

These basic cyclic functions can be transformed as follows.

1. The amplitude can be changed.

The amplitude of y = sin x is 1. The amplitude of y = 3 sin x is three. The amplitude of y = 1/2 sinq is 1/2.



2. **The size of the interval needed to complete one cycle can change**. This is the period of the cycle, and it can be squeezed or stretched. How often the fundamental cycle is completed can be changed. This is the frequency. The frequency in cycles per unit on the x axis equals 1 divided by the period. The coefficient of x divided into 2≠ gives the period.

Examples

1. The period for y = sin x is 2≠. The period (p) of y = sin 2x is p = 2≠/2 = ≠. This means that y = sin 2x will squeeze the fundamental cycle into the interval 0, ≠ on the x axis. It will complete two cycles for every cycle completed by y = sin x. The frequency y = sin 2x is once for every ≠ units. If the units are seconds, the function will complete one cycle every 3.14… seconds.

2. The period for y = sin $\frac{1}{2}$ x is p = $\frac{2\neq}{1/2}$. This means that y = sin $\frac{1}{2}$x will stretch the fundamental cycle over the interval 0, 4≠. The frequency is one cycle for every 4≠ units on the x axis. It is "half as fast" as y = sin x.

3. **Horizontal shifts**: The "starting point" for a fundamental cycle can be changed. These changes are called phase shifts, or **phase angle** changes. These changes displace the curve for y = sin x right or left on the x axis by a constant amount. If a positive number is added to x the shift is left, if a negative number is added, the shift is to the right.

4. **The curve can be shifted vertically** up or down by a constant amount that is added or subtracted from y = sin x. Without changing the amplitude, each y value for the function is moved up or down a constant amount. This amount is called the **displacement**, d. It will move the graph up if the number is positive and down if the number is negative. The line y = amount of displacement (d) will be where the x intercepts of y = sin x are moved to.

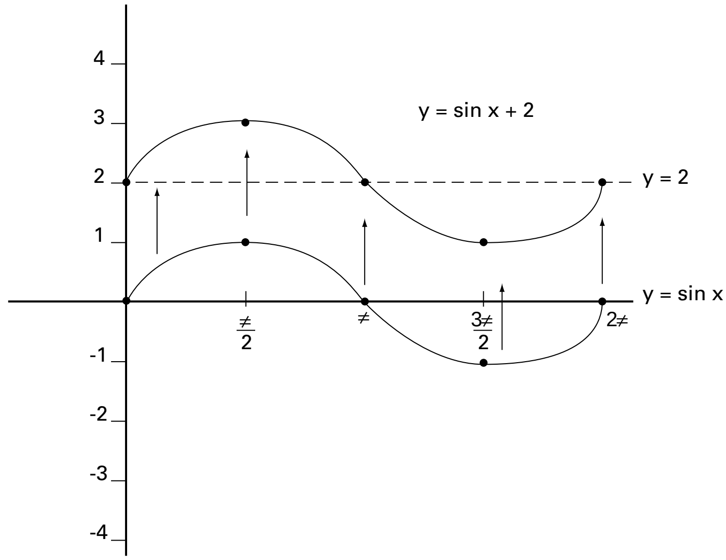
Example

y = sin x + 2 will move the graph of y = sin x up two units on the y axis. The line y = 2 will now be intercepted by the graph of y = sin x + 2 at the same points that were intercepted on the x axis for y = sin x. Refer to the table of key values given earlier for y = sin x to see that each input value for sin x is the same on the graph for y = sin x + 2 but the output values are each increased by 2.

If we let a = the amplitude, and d = the displacement, then the range of values for a sine function will be between - ÍaÍ + d (the minimum value), and Ía + dÍ, the maximum value. Here you can see that y = sin x + 2 ranges from y = 1 to y = 3. Notice that this curve still has an amplitude of 1, there is no change in amplitude in going from y = sin x to y = sin x + 2.

Unit 1 – Equations and Patterns

Topic 2 – Trigonometric Functions



You can sketch the graph of a trigonometric function when you know the fundamental cycle involved and the changes caused by these four features.

A general equation shows how these changes will transform the fundamental cycles of the sine and cosine:

y = a sin k (q + c) + d and y = a cos k (q + c) + d

amplitude = Ía Í

period = $\frac{2≠}{k}$ and frequency = $\frac{k}{2≠}$

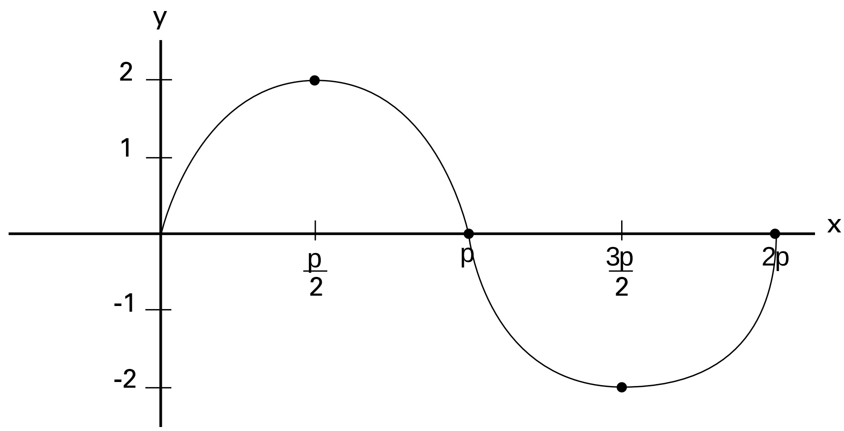
Phase angle shift = Íc Í units to the right for negative c, and left for positive c

Vertical shift = Íd Í units up for positive d, and down for negative d.

When a = 1, k = 1, c = 0, d = 0 we have the graphs for y=sin x and y=cos x. Any other values for these variables will transform the basic sine curve.

Examples

1. What is the equation for this curve?



This is a sine curve with period 2≠ and no phase shift and no vertical displacement. The amplitude is 2. We use the general equation to capture this information: a = 2, k = 1, c = 0 and d = 0. This leaves us with y = 2 sin x as the equation for this graph. (Keep in mind that x and q refer to the same thing: an input number in radians that can be any real number when the sine and cosine are involved. Also note that x or q can be changed into degrees whenever desired.)

2. Find the amplitude, period, phase shift and vertical displacement for y = 2sin(3x + ≠) + 1. Sketch the graph.

Rewrite the equation in standard form by factoring out 3.[16]

$$y = 2\sin\left(3\left(x + \frac{\neq}{3}\right)\right) + 1$$

Now locate the values for the variables in the general equation:

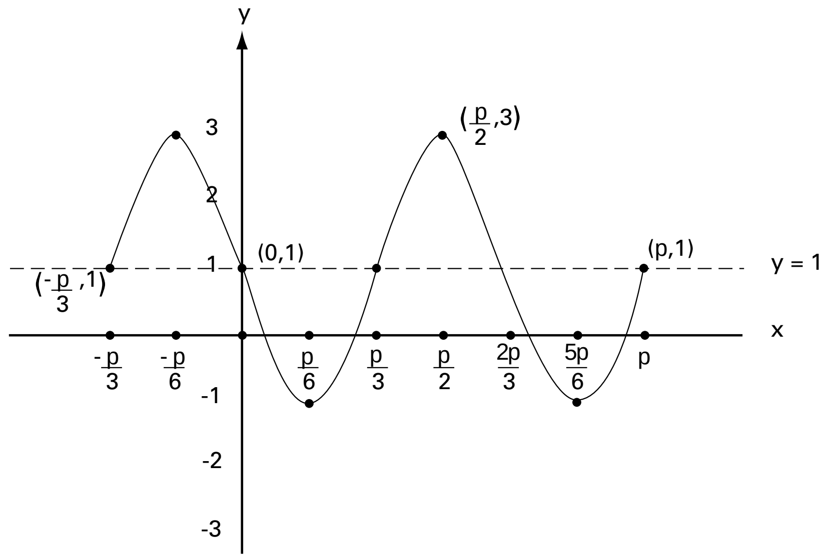
Amplitude = 2

period = 2≠/3
phase shift = ≠/3 to the left
vertical displacement = 1 unit up

This information will allow us to sketch the graph. You can see how the graph of y = sin x was shifted, given a greater amplitude, and squeezed so that one cycle takes only from 0 to 2≠/3 instead of from 0 to 2≠. The line for y = 1 also shows the vertical displacement up one for the x intercept points of y = sin x.

[16] Some textbooks give phase shift = -C/B, if B is not factored out from the form y = A sin (Bx + C) + d. In this curriculum B is factored out to isolate C. If C is positive, the shift is left, if C is negative the shift is right.

Topic 2 – Trigonometric Functions



3. Write the equation for a sine curve that has an amplitude of 3, a period of 2≠, a phase shift of ≠/6 to the left and a vertical displacement of 1.

   First recall the form of the general equation for sin x: y = a sin k(q+c) + d .

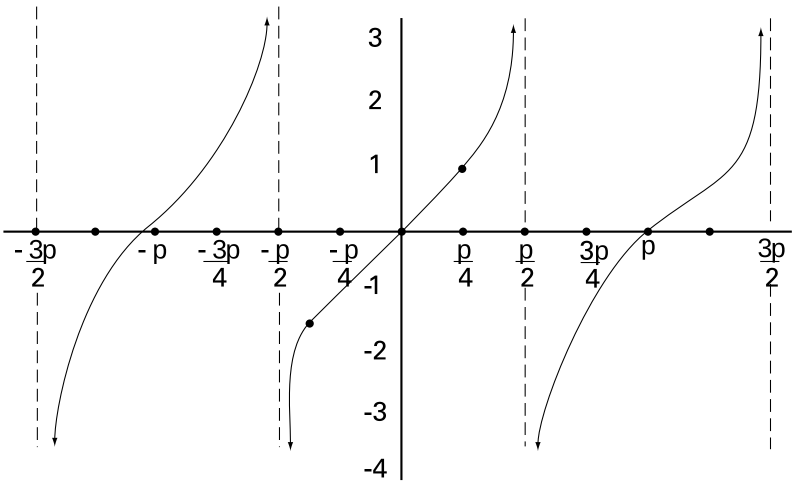
   Replace the variables with the values given in the problem: a = amplitude = 3, period = 2≠/k = 2≠, shift = ≠/6 (shift is left so c = -(-≠/6)), displacement = 1, so points on the curve will move from x intercepts to y = 1 intercepts. The equation is y = 3 sin (x + $\frac{≠}{6}$) + 1

Topic 2 – Trigonometric Functions

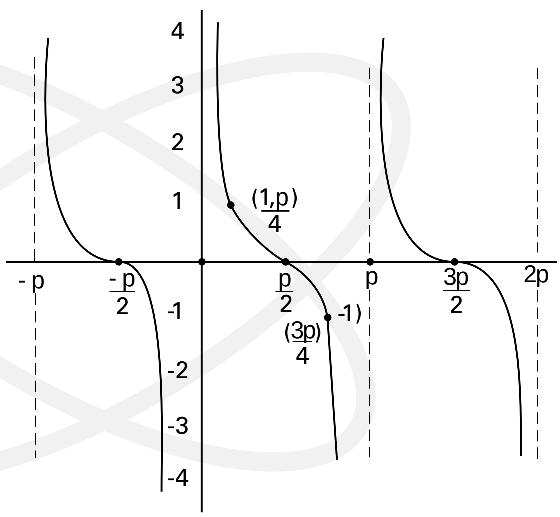
The Tangent Function

The tangent function is undefined whenever the cosine function equals zero. Recall that the tangent is the sine divided by the cosine, that division by zero is undefined, and that cosine equals zero at ≠/2, 3≠/2 and all integer multiples, ≠/2 + k≠, and 3≠/2 + k≠. The fundamental cycle of the tangent function is ≠. There are five key points that help us to sketch the graph of tangent x :

x	-≠/2	-≠/4	0	≠/4	≠/2
Tan x	undefined	-1	0	1	undefined



The lines where x = ≠/2 and its multiples are **asymptotes**. As x approaches ≠/2, the tangent of x goes towards infinity. Test this on your calculator by entering 89°, 89.5°, 89.9°, 89.99°. Notice how rapidly the tangent increases as you get closer to 90 degrees.



y = cot x

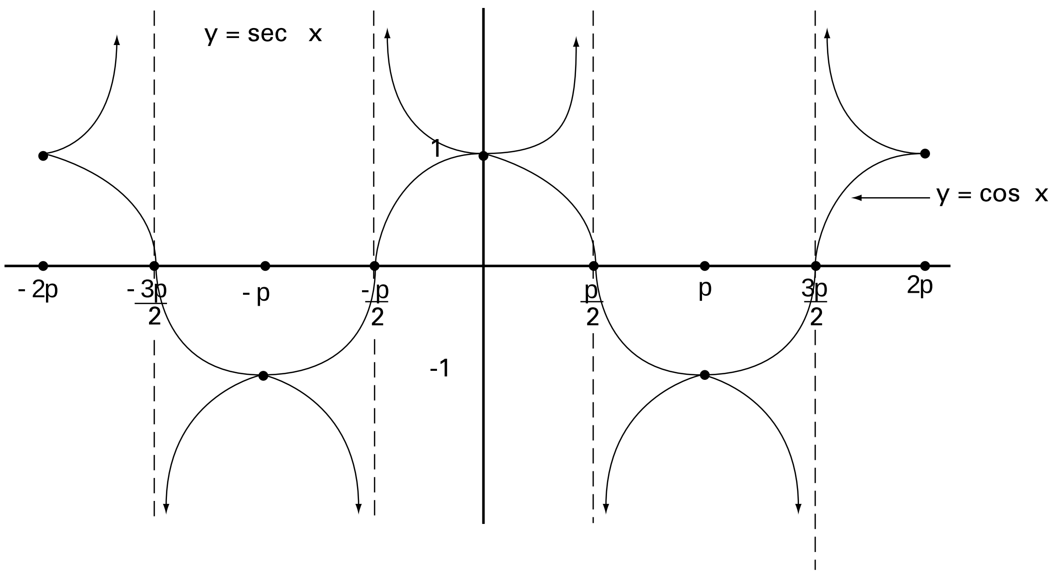
x	0	≠/4	≠/2	3≠/4	≠
coT x	undefined	1	0	-1	undefined

Topic 2 – Trigonometric Functions

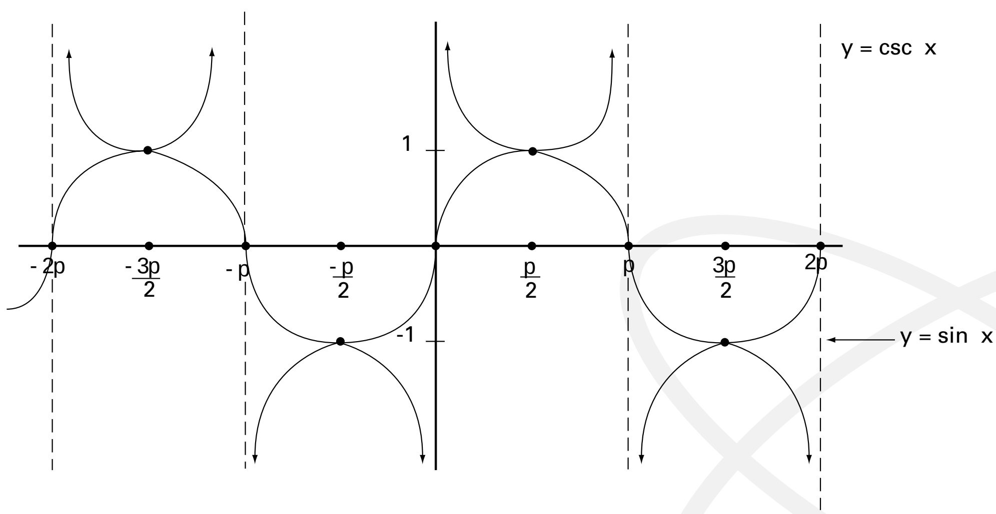
This diagram shows the reverse direction of the cotangent function.

Transformations of the tangent and the remaining three trigonometric functions are based on the fundamental cycles of sine, cosine and tangent.

The graphs of the secant and cosecant are related to the graphs of the cosine and sine respectively. The cotangent is related to the tangent.



The period of the secant is 2≠. The asymptotes are parallel to the y axis at x = ≠/2 and every multiple of ≠/2 + ≠ in both directions. These are the x intercepts of y = cos x. There is no maximum value , so the secant has no amplitude and approaches infinity as x approaches but never reaches each asymptote.



The period of the cosecant is 2≠ . The asymptotes are at x = ≠ and every multiple of ≠ in both directions. These are the x intercepts of y = sin x.

Topic 2 – Trigonometric Functions

Graphing Tips: Use The Five Key Coordinates

and:

1. Label the x axis with a coordinate = phase shift and find y
2. Label the coordinate equal to the phase shift + one period and find y
3. Label the midpoint between the phase shift and the end of one period
4. Label the points midway between the center and the extremes.

Sample Problem

1. Sketch the graph of $y = 4\sin\left(2x - \frac{\neq}{3}\right)$

Step one:
Factor out 2 to get the equation into standard form: $y = 4\sin 2\left(x - \frac{\neq}{6}\right)$

Step two:

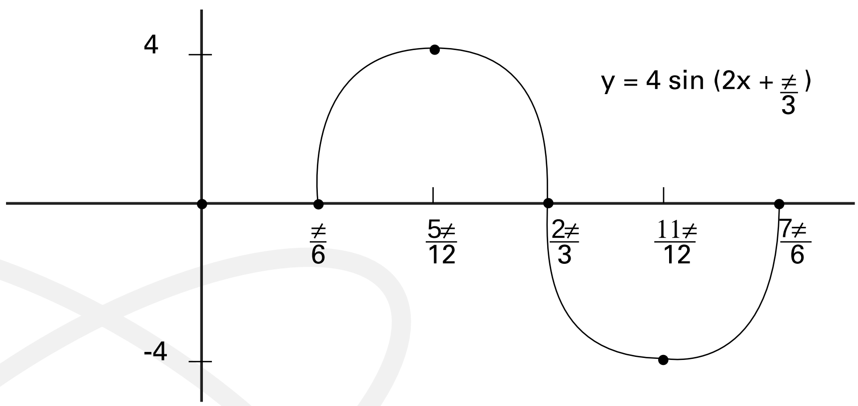
Identify amplitude, period , phase, and displacement.
$A = 4$, $p = 2\neq/2 = \neq$, phase = $\neq/6$ right, displacement = 0.

Step three:

Mark the period on the x axis starting with the phase shift $\neq/6$ right of the origin and ending at $\neq/6 + \neq = 7\neq/6$. At these points and at the midpoint the equation equals 0.

Step four:

Find the midpoint between $\neq/6$ and $7\neq/6 = 2\neq/3$, and then between the midpoint and the extremes of the period, $5\neq/12$, and $11\neq/12$. Mark $y = 4$ at $5\neq/12$ and -4 at $11\neq/12$.



Topic 2 – Trigonometric Functions

Trigonometric Identities

Identities help us to simplify expressions and solve equations involving the trigonometric functions.[17] An identity is true for all numbers that both sides are defined for. You have already worked with the definitions of six ratios that define the trigonometric functions. These are also examples of identities.

To prove that an identity is true, work on one side and try to get it to equal the other side by substituting proven identities for parts of the expression. Don't try to simplify both sides as you would an equation.

The Fundamental Identity

Because the x coordinate of any point on the unit circle is the cosine of the central angle corresponding to that point, and because a right triangle can be formed with x and y as sides and the radius of one as the hypotenuse, we can use the Pythagorean theorem to conclude in general that

$$\sin^2 q + \cos^2 q = 1.$$

This relationship is true for all points on the circumference of a unit circle. Any point on the circumference can be described as a trigonometric point. A trigonometric identity is true for all values of q. Notice that this identity can be rearranged and related to the definition of the tangent to give other useful **Pythagorean identities:**

$\sin^2 q = 1 - \cos^2 q$

and

$\cos^2 q = 1 - \sin^2 q$

and

$1 + \cot^2 = \csc^2 x$

$\tan^2 x + 1 = \sec^2 x$

Sum and Difference Identities

$\cos(a + b) = \cos a \cos b - \sin a \sin b$

$\cos(a - b) = \cos a \cos b + \sin a \sin b$

Double Angle Identities

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

17 Practice working with identities using a college algebra and trigonometry textbook. Several are included in the list of resources for this curriculum

Topic 2 – Trigonometric Functions

Examples

1. What is the exact value of cos 75°?

   Use the identity for cosine of the sum of two angles and pick two angles for which we have exact values:

   $\cos 75° = \cos(30° + 45°)$

   $= \cos 30° \cos 45° - \sin 30° \sin 45°$

   $= \left\{ \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} \right\} - \left\{ \frac{1}{2} \times \frac{\sqrt{2}}{2} \right\}$

   $= \frac{\sqrt{3}\sqrt{2}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{2}\;(\sqrt{3}-1)}{4} = .2588 = \cos 75°$

   Check this result with your calculator.

2. Verify the identity $\frac{\csc x - \sin x}{\sin x} = \cot^2 x$

   The left side can be changed into the right by rewriting it to use the fact that csc x = 1/sin x:

   $\frac{\csc x - \sin x}{\sin x} = \frac{\csc x}{\sin x} - \frac{\sin x}{\sin x}$

   $= \csc x \; X \; \frac{1}{\sin x} - 1$

   $= \csc x^2 - 1$ (use Pythagorean identity)

   $= \cot^2 x$

3. Use an identity to find tan(120°)

   Because we know the tangent of 60° exactly, we can use the identity for tan 2x by letting x = 60°:

   $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

   $\tan(120°) = \frac{2 \tan 60°}{1 - \tan^2 60°} = \frac{2\sqrt{3}}{1 - (\sqrt{3})^2} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$

Practice makes perfect when it comes to working with trigonometric identities. A big part of practice is "trial and error" using the basic identities given in this topic.

Unit 1 – Equations and Patterns

UNIT 1

Topic 2 – Trigonometric Functions

Trigonometric Equations

Trigonometric equations have solutions that can be restricted to the unit circle, or that can include these + k2≠, where k is a positive integer. Identities can be used to express one trigonometric function in terms of another function. Before an equation can be solved, it must contain expressions using only one function. The tools of algebra, including the quadratic formula, are used to solve trigonometric equations. In work with trigonometric equations care must be taken to find all possible solutions unless instructions are given that restrict the domain.

For example one solution to the equation sin x = 1 is ≠/2. This is the only solution between 0° and 360°. However, If we are asked to find the general solution for sin x = ≠/2 for all x, we must include ≠/2 + all integer multiples of 2≠. ≠/2 + 2≠, ≠/2 + 4≠, …≠/2 + 6≠ , etc. are also solutions for x in sin x = 1.

A More Detailed Explanation

Recall that ≠/2 is the arc length s, swept out by the radius vector between (1,0) and (0,1) on the unit circle. the y coordinate of the point (0, 1) is the sine of ≠/2, so sin ≠/2 = 1. This is the only point on the unit circle with a y coordinate of 1.

An infinite number of other solutions also exist based on this "simplest" , or smallest non-negative solution on the unit circle. Every integer multiple of 2≠ that is added to ≠/2 will produce another solution because that number will produce a co-terminal angle and have a sine equal to 1 as well. The wrapping function discussed earlier also shows this. All of the angles that are co-terminal with ≠/2 will have the same sine. The general solution to the equation sin x = 1 is written this way:

$$\left\{ x \;\middle|\; x = \frac{\neq}{2} + k2\neq \right\}$$

Read it this way: " the solution set for sin x = 1 contains all x such that x = $\frac{\neq}{2}$ or any positive integer multiple of 2≠ added to ≠/2." (Note that k represents any positive whole number). It is important to understand that the general solution for a trigonometric equation will not only identify a coordinate of a point on the unit circle, but also include all multiples based on additional revolutions around the unit circle. Every revolution around the circle produces another real number, that solves this equation.

To Find the General Solution for a Sine or Cosine Equation:

1. Isolate the function you want a solution for
2. Simplify the resulting expression it is equal to (identities may be helpful)
3. Find the angles (or numbers in radians) for 0 ″ x ″ 360° that satisfy the function. Use the inverse function key on a calculator, use a table, or use an exact value you have learned.
4. Express the answer as the angle(s) found in 3 above plus 2k≠ multiples of it.

Unit 1 – Equations and Patterns

Topic 2 – Trigonometric Functions

Examples

1. What is the general solution for 2 cos x = .75 ? What is the sine of x in the first quadrant?

   2 cos x = .75

   cos x = .375 (divide both sides by 2 to isolate cos x) (step 1)

   .375 cannot be simplified further (step 2)

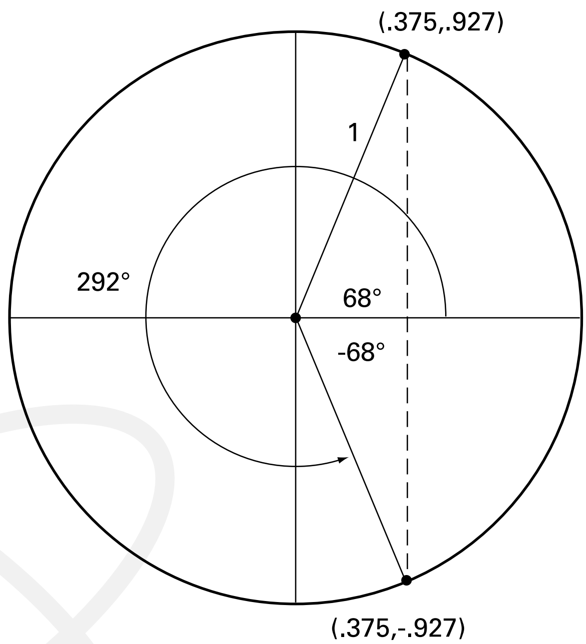
   We now have to find all of the values for x that have a cosine of .375.

   The solution will be the x coordinate of all points on the circumference of a circle such that cos q = $\frac{x}{r}$ = .375.

   The value of q can be found by looking at a table or using a calculator with the inverse function $\cos^{-1}$. $\cos^{-1}$ .375 = 1.186, or 68°. -68° is co-terminal with 292° and will also have a cosine of .375. (step 3)

   A vertical line connects the two trigonometric points. The general solution will be 68° in the first quadrant or any multiple of 2≠ + 68°, and 292° + 2≠k in the fourth quadrant. (step 4)

   Note: your calculator will allow you to change from radians to degrees and vice versa with a "DRG" or similar key.



The Pythagorean theorem allows us to find y = .927. You can see that the sine of 68° is positive, and the sine of 292° is negative. Sin x in the first quadrant = sin 68° = .927.

2. What is the solution for sin x = -1?

   On the unit circle, y = -1 at 270° = 3≠/2 . Sin 3≠/2 = -1. No other values for q will have a sin of -1. The general solution will also include all multiples of 2≠ added to 3≠/2. The complete solution is:

$$\left\{ x \,\middle|\, x = \frac{3\neq}{2} + k2\neq \right\}$$

3. What is the solution for 1/2 sin x = 20

   We solve for sin x and get sin x = 10, but this is impossible because the maximum value of sin x is 1. This equation has no solution.

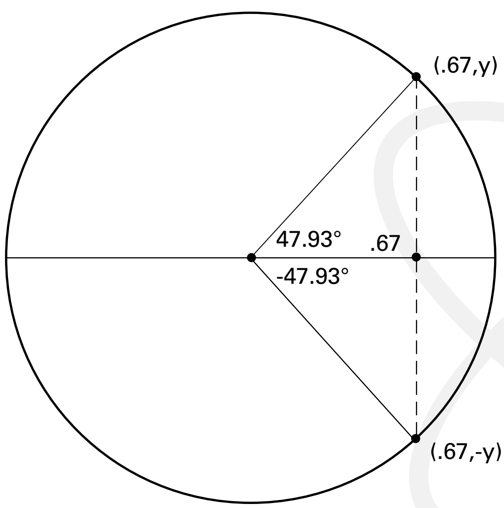
4. What is the general solution for cos x = 1?

   A look at the unit circle shows that x = 1 at q = 0°, 360°, and multiples of 360°. You can also see that cos x = 1 only when the coordinates for the terminal point on the arc length corresponding to q are (1,0). The solution set is:

$$\left\{ x \,\middle|\, x = 2k\neq \right\}$$

5. Find x when cos x = .67

   Here we need to find the numbers on the unit circle that have a cosine of .67. You can see that .67 is the x coordinate of the trigonometric points corresponding to x. There will be two points on the unit circle that have this x coordinate (i.e. that have the same cosine) and they are connected by a chord as shown in the next diagram. Use the inverse cosine key on your calculator to get $\cos^{-1}$ .67 = .836 = 47.93°. This is one solution, and the other solution will be –47.93°, which is co terminal with 312.07°. The remaining solutions are multiples of 2k≠ added to each of these two "simplest" solutions on the unit circle.



This problem shows us that when the cosine of a number, a, is between –1 and 1, the solution will always involve two points joined by a vertical chord. The solution will have this form:

Topic 2 – Trigonometric Functions

$\{x \text{ Í} x = \cos^{-1} x + 2k\neq, \text{ or } x = 2\neq - \cos^{-1} x + 2k\neq\}$

In this example the solution is

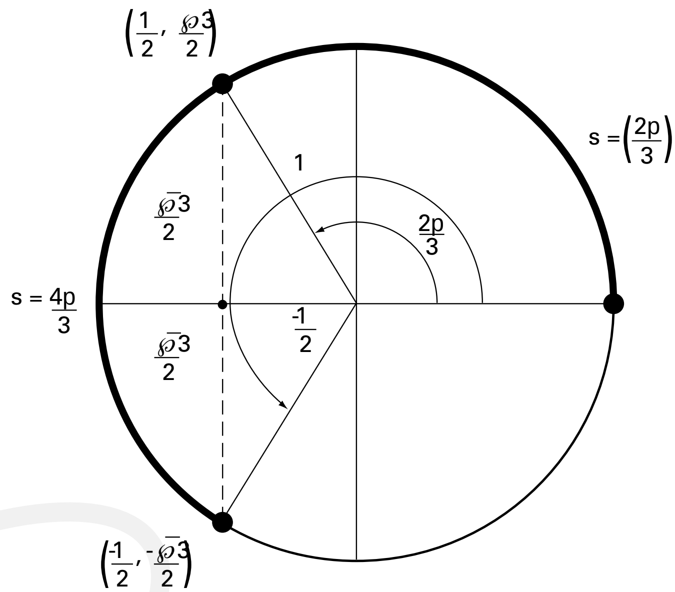
$\{x \text{ Í } x = 47.93° + 2k\neq, \text{ or } x = 2\neq - 47.93° + 2k\neq\}$

6. Solve for x: cos x = 3.14

   It is impossible for the x coordinate of any trigonometric point to be greater than one, so this equation has no solution.

7. Find the solution for cos x = -1/2

   Sketch the problem on the unit circle. x = -1/2 for angles in quadrants 2 and 3. a chord can be drawn connecting the two trigonometric points that have an x coordinate of -1/2. You know from earlier work that 2≠/3 is the arc length that terminates in a point with -1/2 as its x coordinate. This fact can be recovered from a sketch showing the right triangle that has a base of 1/2 and a hypotenuse of 1.

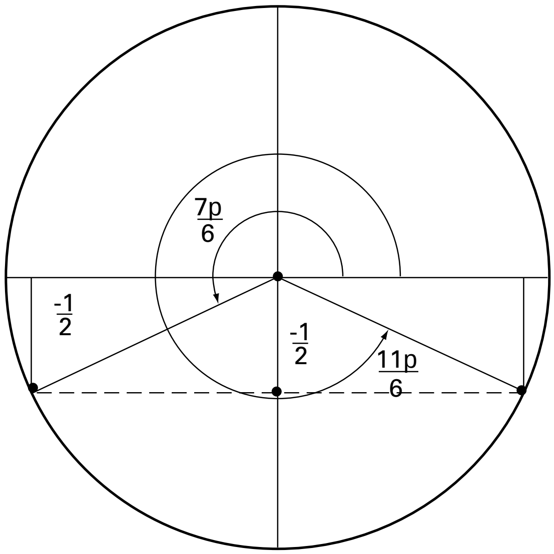


The general solution will include all multiples 2k≠ of the angles corresponding to the two points on the unit circle that have an x coordinate of -1/2. The solution is

$$\left\{ x \,\middle|\, x = \frac{2\neq}{3} + 2k\neq \text{ or } x = \frac{4\neq}{3} + 2k\neq \right\}$$

8. Find all real numbers such that sin x = -1/2

   This problem is similar to #7 above, except that the chord joining the two solutions on the unit circle will be horizontal (parallel to the x axis). You should be able to picture the simplest solutions mentally. The simplest solutions will be the angles corresponding to the trigonometric points with a y coordinate = -1/2 on the unit circle. The reference angle for x is 30°, and the negative sign tells us that x has its terminal sides in quadrants 3 and 4. These are the terminal points of 7≠/6 + 2k≠, and 11≠/6 + 2k≠.



   the solution will be

$$\left\{ x \,\middle|\, x = \frac{7\neq}{6} + 2k\neq \text{ or } x = \frac{11\neq}{6} + 2k\neq \right\}$$

9. Solve for q in 2 sin q - 1 = 4 sin q - 2, for 0 ″ q ″ 2≠.

   Group sin q on one side and combine like terms:

   2 sin q = 1

   sin q = 1/2

   The general solution was given above in #8 for sin x = -1/2. The solution for sin q = 1/2 is similar with the horizontal chord joining trigonometric points in the first and second quadrants where the sine is positive. Since we are asked only for solutions on the unit circle (no multiples of 2≠) the answer is 30° and 150°. However, we are given the domain in radian measure, therefore the answer should be in radians: ≠/6, and 5≠/6.

Topic 2 – Trigonometric Functions

Second Degree Trigonometric Equations

The methods of algebra combined with identities and the quadratic formula allow us to solve second degree equations. The domain will be specified.

Examples

1. What is the solution of $\sin^2 x + 2 = 4$

   Solve for sin x: $\sin x = \sqrt{2}$

   The general solution will be all angles with $\sin x = \sqrt{2}$, however $\sqrt{2}$ is greater than one and therefore this equation has no solution.

2. What is the solution for q in 0 ″ q ″ 2≠ in the equation $2\cos^2 q - \cos q = 3$?

   This equation can be factored after it is set equal to zero (i.e. put into standard form). Each factor is then set equal to zero. Only the solutions that are defined by the cosine function will be valid.

   $2\cos^2 q - \cos q = 3$

   $2\cos^2 q - \cos q - 3 = 0$

   $(2\cos q - 3)(\cos q + 1) = 0$

   $2\cos q = -3$

   $\cos q = \frac{3}{2}$

   $\cos q = -1$

   Only the second factor produces a solution because the cosine cannot exceed +1 or be less than -1. The solution on the unit circle is 0°, and 360° or 2≠ where cosq = 1.

3. Find the solution for $4\sin^2 q + 3\sin q - 1 = 0$ between 0 and 2≠ inclusive.

   This problem does not factor. It is already in standard form and we use the quadratic formula.[18] The coefficients are a = 4, b = 3, c = -1.

   $4\sin^2 + 3\sin q - 1 = 0$

   $$\sin q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$\sin q = \frac{-3 \pm \sqrt{25}}{8}$$

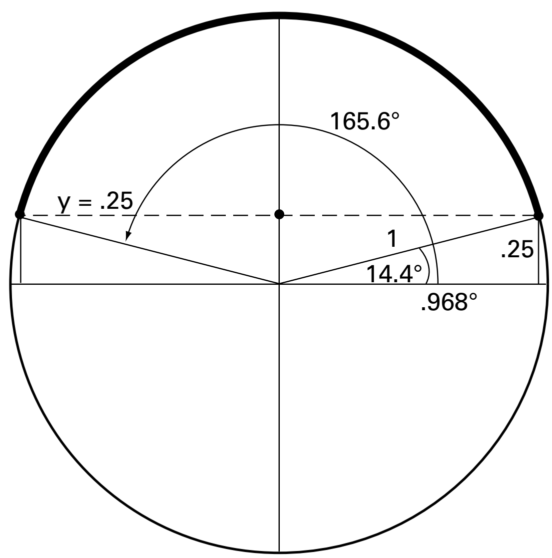
   $$\sin q = \frac{-3 + 5}{8} \text{ or } \frac{-3 -5}{8}$$

   $\sin q = .25$ or -1

18 The Quadratic formula is reviewed in the supplemental topics for Math – Module 5.

Both sets of solutions are meaningful and fall within the bounds of -1, 1. To find q, use a calculator or a sketch of the unit circle. When sinq = -1, q = 3≠/2, there are no other angles that have a sine of -1 on the unit circle. We do not include multiples of 2≠ in the solution because the problem only asks for solutions between 0° and 360°.

The second set of solutions can be found by using a calculator to find $\sin^{-1}$ .25 = 14.47°, or .253 radians. A horizontal chord connects the point with this y coordinate to the point in the second quadrant that determines another central angle with a sine of .25. Note that the Pythagorean formula allows us to find x. The second value for q is 180° – 14.4° = 165.6°, or about 2.89 radians. Since we are only asked for solutions between 0 and 2≠, this completes the task. The solutions are 3≠/2, .253, and 2.89. In degrees these are 270°, 14.47°, and 165.5°



Topic 3 – Exponential Functions

Review the properties for exponents given in Math foundations for this topic. Exponential functions have the form $y = a^x$. **The exponent is the variable and the base is constant.** The base for all equations that have this form is restricted to a greater than zero and a not equal to one. If the base were allowed to be negative, the function would not describe a real number. For example $(-2)^{1/3}$ is not a real number. Also, 1 is excluded because it would produce the line y = 1, a constant function.

Exponential Growth Is Rapid

Contrast the exponential function to the algebraic function $y = x^a$ where the base is variable and the exponent is constant. In both functions x can be any real number.[19] Exponential functions describe patterns found in nature that are very much "faster" than algebraic curves of the form $y = x^a$. The growth of populations (including continuous compounding of an investment) and the decay of radioactive materials follow this pattern. In these cases the amount at a given time is proportional to the starting amount.

Use Technology To Analyze Functions

Compare some values for $y = 2^x$ and $y = x^2$ on your calculator to see the difference in how fast they increase. For example, when x = 20, use the x^y key to find $2^{20} = 1{,}048{,}576$ (keystrokes: 2, x^y, 20, =). This is the value of the exponential function with base 2 raised to the 20th power.

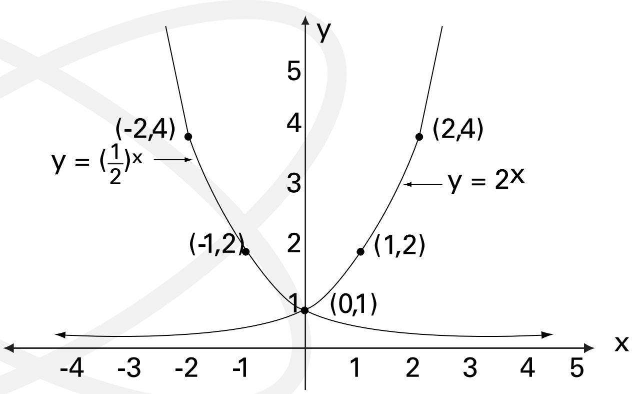
Now try 20 as the input for the algebraic function $y = x^2$. $20^2 = 400$. (keystrokes: 20, x^2, =)

You can also use a graphing calculator to sketch these curves.

The next diagram shows the properties of exponential functions with 0 < a < 1 in

$y = \left(\frac{1}{2}\right)^x$ and a >1 in $y = 2^x$.

This graph illustrates some properties of all exponential functions that you need to know.



x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$		1	2	4	8

x	-2	-1	0	1	2	3
$\left(\frac{1}{2}\right)^x$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

[19] Irrational numbers are values admissible for x. Rational approximations can be found for a raised to any irrational power. The limit of these approximations is the value of the exponential function. For example $2^{\sqrt{3}} = 3.321997085$ calculated to 11 places on a calculator.

Topic 3 – Exponential Functions

Properties of Exponential Functions

1. The point (0, 1) is on both graphs. This is the y intercept of all exponential functions of this form. If the function is transformed (shrunk, stretched, reflected) then the y intercept will change. For example $y = -4 + 3^{x+2}$ is a transformation of $y = 3^x$. If you sketch $y = 3^x$ you will get a graph similar to $y = 2^x$ with a y intercept of (0,1). This graph will shift 2 units left and 4 units down to produce the graph of $y = -4 + 3^{x+2}$. See next example.

2. If the base is greater than one the function will increase towards infinity as x increases and the negative x axis will be an asymptote. These functions are called increasing exponential functions. Example: $y = 2^x$. As x approaches $-\infty$, 2^x will approach 0 but never reach it.

3. If the base is between 0 and 1, the function will decrease towards 0 but never reach 0 as x increases. These functions are called decreasing exponential functions. This means that the positive x axis will be an asymptote. As x decreases towards $-\infty$, $\left(\frac{1}{2}\right)^x$ increases towards positive ∞. Example $y = \left(\frac{1}{2}\right)^x$

4. If $a^{x_1} = a^{x_2}$ then $x_1 = x_2$. The exponents will be equal when two exponential functions with the same base equal each other. As stated earlier, this is true for $a > 0$ and $a \neq 1$. **This one to one property is the key to solving many exponential equations and will be used in the examples below.**

Graphing Exponential Functions

You can graph an exponential function by starting with the point (0,1) for the form $y = a^x$ then sketching a few points chosen for convenience depending on whether a is between 0 and 1, or greater than one. This will give you a reference equation.

Next look at the equation you are graphing and identify any vertical or horizontal shifts. Relate these shifts to the reference equation you are using. Identify the new asymptotes if there is a horizontal or vertical shift. Select a few values for the equation to help identify points on it for accuracy. A graphing calculator can be used or you can combine a table with your knowledge of transformations to the basic form of the exponential function. The next examples review what you need to know about transformations to exponential equations.

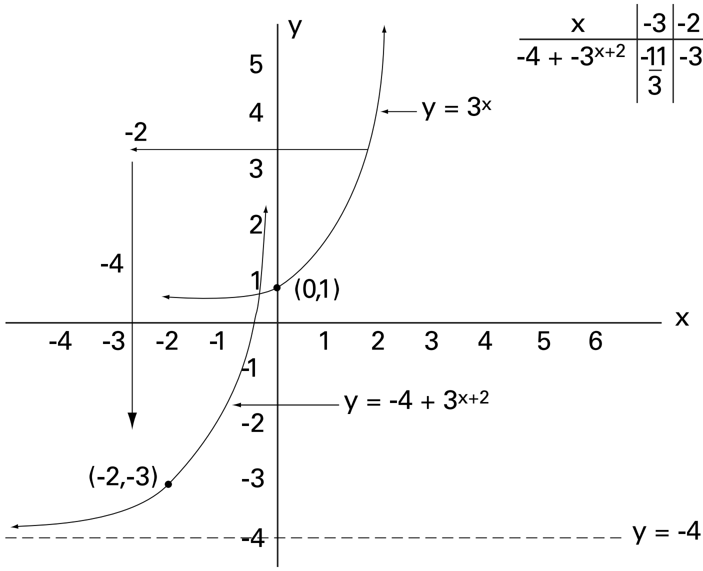
Examples

1. Graph $y = -4 + 3^{x+2}$

   Step One: recognize that this is a transformation of $y = 3^x$. In general you can relate any exponential equation to the form $y = a^x$. Transformations to this form can involve adding or subtracting a value to the exponent or to a^x, or multiplying a^x by a factor. In this example 2 is added to the exponent, and -4 is added to a^x .

Step Two: look at the exponent. A number added to x indicates a horizontal shift of the x coordinate of each point. If the number is positive the shift is left, if negative the shift is right.

Step Three: look at the number added to the basic form. In this example -4 is added to 3^{x+2}. If the number is negative, the y coordinates of $y = 3^x$ are shifted down, if positive they are shifted up. In this example the shift is down 4 units. The line $y = -4$ is the asymptote for $y = -4 + 3^{x+2}$ as x decreases instead of the x axis.

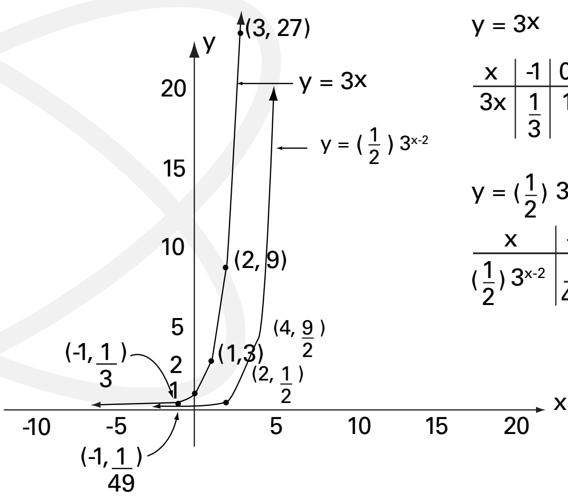


x	-3	-2	-1	0
$-4 + -3^{x+2}$	$-\frac{11}{3}$	-3	-1	5

In this diagram you can see how the graph of $y = 3^x$ was shifted left 2 units and down 4 units to give the graph of $y = -4 + 3^{x+2}$.

2. Sketch the graph of $y = \left(\frac{1}{2}\right)3^{x-2}$. Indicate the domain, range, the horizontal asymptote, and whether the function is increasing or decreasing.

The -2 in the exponent will shift the graph of $y = 3^x$ two units right. Multiplying by 1/2 will have the effect of making the curve less steep, it will increase to infinity more "slowly" as the tables in the diagram show.



$y = 3^x$

x	-1	0	1	2	3	4
3^x	$\frac{1}{3}$	1	3	9	27	81

$y = \left(\frac{1}{2}\right)3^{x-2}$

x	-1	0	1	2	3	4	5
$\left(\frac{1}{2}\right)3^{x-2}$	$\frac{1}{49}$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	$\frac{27}{2}$

Notice that the graph for $y = \frac{1}{2}3^{x-2}$ is below the graph for y = 3x along the negative x axis. Both functions are increasing exponential functions (i.e. as x increases so does y in both equations.) Both have the negative x axis as their horizontal asymptote.

Exponential Equations

By using the one-to-one property between bases and exponents, we can solve equations by using algebra to isolate the exponent that is the variable in an exponential equation. The properties of exponents used in this process were reviewed in math foundations.

> **Express Both Sides Of The Equation In Terms Of Powers Of The Same Base.** Then the exponents can be set equal to each other.
>
> If $a^{x_1} = a^{x_2}$ then $x_1 = x_2$
>
> for $a > 0$ and a ↑ 1

Examples

1. $\frac{1}{8} = 2^x$

   $\frac{1}{2^3} = 2^x$

   $2^{-3} = 2^x$

   $-3 = x$

   Here we express 1/8 as a power of the base 2.

   Check: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

2. $\left(\frac{1}{10}\right)^x = 10{,}000$

   Here we need to express both sides as a power of 10 or of 1/10.

   $\left(\frac{1}{10}\right)^x = 10{,}000 = 10^4$

   $10^{-x} = 10^4$

   $x = -4$

   check: $\left(\frac{1}{10}\right)^{-4} = 10^4 = 10{,}000$

3. Find the value for x that will make $y = 4^{2-x}$ equal 64

   In this problem we are given the y value (an element in the range) and asked for the corresponding x in the domain. We know there is only one such value because the exponential function is "one to one". 64 can be expressed as a power of 4.

   $64 = 4^{2-x}$

   $4^3 = 4^{2-x}$

   $3 = 2 - x$

   $x = -1$

   Check: $4^{2-(-1)} = 4^3 = 64$

The Base e

The irrational number e @2.71828… is stored in your calculator because it turns out to be an important base for exponential functions.[20] When a = e we have a base e exponential function. $y = e^x$ is a very important form for many useful equations. The graph of $y = e^x$ is very close to the graph for $y = 3^x$ because 2.718.. is close to 3. When a quantity changes at a rate directly proportional to the amount present at a given time t, the amount of the quantity is an output of some version of the exponential growth/decay function.[21]

> **An equivalent logarithmic form can be found for every exponential equation and vice versa as you will see in the next topic.**
>
> We will discuss logarithms in the next topic, but it is a good idea to realize here that **the exponential function has the logarithmic function as its inverse. By definition $\log_a y = x$ and $a^x = y$.** x is the number (i.e. the exponent) that a is raised to in order to produce y. A logarithm is an exponent. The log of y to the base a is x. The base e is used so often for the value of a, that In is the abbreviation for "natural logarithm" that is used to mean a log with base e. In e = 1 because $e^0 = 1$, and In 1 = 0 because $e^0 = 1$. Your calculator will have a key that gives In x, and an inverse that gives e^x. You will use the In x key in sample problem two below.

Example

1. When money is invested at a fixed annual rate for a period of time, then the balance at any given time will be

   $$A = P \times e^{rt}$$

   where P is the starting amount, A is the amount after t years have gone by, and r is the annual rate compounded continuously.

[20] On your calculator press e^x, then enter 1 to get the value stored for e. e is the limit of a series that describes compounding.

[21] Using methods from calculus it can be shown that $\left(1+\frac{r}{n}\right)^{nt}$ approaches e^{rt} as $n \to \infty$.

If $7500 is invested at a 5% annual rate that is compounded continuously, what will the balance be after 4 years and 6 months?

To use the formula, change 4 years and six months into 4.5 years, set r =.05, t= 4.5, and P = $7500. Then:

$A = \$7500 \times e^{(.05 \times 4.5)}$

$A = \$7500 \times e^{.225}$ (Use the ex key to find $e^{.225}$.)

$A = \$7500 \times 1.2523 = \$9392.42$

The same formula is used to find the amount left in a radioactive decay process or any other declining population problem. Notice that the concept of a "population" means any quantity that can be counted. Changes to a population are changes to a finite quantity. In the case of radioactive decay, (or any declining population problem) the rate will be negative since the amount of remaining radioactive material is continuously growing smaller. Sample problem two below illustrates a negative growth rate.

Sample Problems

Optional: taking the limit of a step function

Notice that the same deposit would be worth less if the formula for stepwise compounding using annual, monthly, or daily compounding intervals were used instead of the exponential function. The formula for step wise compounding is $A = P\left(1 + \frac{r}{n}\right)^{nt}$. This describes the process of taking a sum of money and adding the interest it earns to the principal at regular intervals. For example, $1000 at 5% annual interest grows to $1050 after a year, and $1102.50 after two years. In the formula, A = the balance after t years, n is the number of regular periods that interest is calculated and added to the previous balance, and r is the fixed rate of interest. It can be shown that as the number of compounding periods increases (i.e. as $n \to \infty$), $\left(1 + \frac{r}{n}\right)^{nt}$ approaches e^{rt}.

Test this with increasing values for n on your calculator.

1. The environment department has kept records and found that the population of beaver in a lake near Fort Smith in any given year (t) is approximately

$$p(t) = \frac{2000}{1 + e^{-.5544t}}$$

Use this exponential equation to find the population today, i.e. when t = 0. Then forecast the approximate population five years from now.

Solution:

This is an equation that uses the base e to calculate continuous changes to the growth of a starting population under stable conditions. The population when no time has elapsed, ie. when we consider the starting population as of today, means that t = 0. Plug in this value for t and calculate the result mentally. The exponent for e will become 0 and $e^0 = 1$. This makes the denominator =2 and then p(t) = 1000. The population now = 1000.

This value may not reflect the facts in this lake, but it provides a basis for forecasting that can be adjusted based on actual observations of the current population. We can formulate the second part of the question more precisely as a hypothetical question: "What will the population be in five years if we have a starting population of 1000 today?" Actual measurements can then be scaled accordingly to this base figure and $p(t_0)$ would equal the beaver population measured today.

To find the population five years from now we set p(t) = 5 and solve for p(5):

$$p(5) = \frac{2000}{1 + e^{-.5544(5)}}$$

$$p(5) = \frac{2000}{1 + e^{-2.772}}$$

$$p(5) = \frac{2000}{1 + .062536\ldots}$$

$$p(5) = 1882$$

There will be approximately 1882 beavers five years from now under stable conditions if we have a starting population 0f 1000 today.

2. One of Issac Newton's contributions to physics was his mathematical description of cooling processes.

$F(t) = T_0 + Ce^{-kt}$

This exponential equation says that the final temperature of a cooling body will equal the starting temperature of the environment it is placed in plus Ce^{-kt} where C and k are constants related to the properties of the body and its environment, and t is the elapsed time after the body has been introduced into an environment with temperature = T_0. Cooling is an example of a negative growth rate, and the constant k is negative to reflect this.

The formula says that the rate at which a substance cools is proportional to the difference in temperature between the body and the temperature of its surroundings. The constants in the formula reflect the characteristics of the bodies involved and a series of temperature measurements in an experiment will give the values for C and k. In the next problem we have to use the information that is provided to find the values of C and k in order to predict the final temperature of a cooling substance.

Question

What will be the temperature of boiling water (= 100° Celsius) be 96 minutes after it is placed in a freezer that is at 0° Celsius? A measurement is taken after 24 minutes and the temperature of the water is found to be 50° Celsius.

Solution

We need to find the value of the constants, C, and k, in this situation. One way is to set F(t) = 0, i.e. to find the value of C before any time has elapsed. This strategy will then allow us to solve for k by using the information given for the temperature taken after 24 minutes. Finally, with both C and k solved, we can plug in the values for t = 96 minutes and find F(96). Note: we will change minutes into hours. 96 minutes = 1.6 hours, and 24 minutes = .4 hours.

The following equations are the result of this approach:

1. $T_0 = 0°$ because we are told that the environment the water is put into is a constant one with a temperature of zero. (i.e. the freezer's temperature doesn't change as the boiling water cools)
2. The value for F(t) at t = 0 is the temperature of the boiling water before any cooling has occurred. Therefore:

   $100° = 0 + Ce^{-k(o)}$

   $100° = Ce^{°} = C$

   We now have a value for C to use.
3. Next find k by using C and the information given about the temperature after .4 hours has gone by.

   $50 = 0 + 100e^{-k(.4)}$

   $.5 = e^{-.4k}$

   $\ln .5 = \ln e^{-.4k}$

   $\ln .5 = -.4k$

   Use the ln key and enter .5 to find x in $e^x = .5$

   $\frac{-.693147}{-.4} = k$

   $k = 1.73286$

With x = .5, we used the inverse of e^x, i.e. ln x, to express .5 as $e^{-.693147}$
Since both sides of the equation now share the base e, we set their exponents equal to each other and solve for k. Check the value for ln .5 = -.693147 by calculating $e^{-.693147} = .5$ on your calculator. This verifies the inverse relationship between ln x = y and $e^y = x$.

4. Now use the values found for C and k to solve the next equation and find the temperature of the water after being in the freezer for 1.6 hours.

   $F(1.6) = 0 + 100e^{-1.73286(1.6)}$

   $F(1.6) = 100e^{-2.772576}$

   F(1.6) = 100 x .0625 = 6.25

   The water will cool down from 100° to 6.25 degrees after being in the freezer for 1.6 hours.

Unit 1 – Equations and Patterns

Topic 3 – Practice Exam Questions

Question 1

What is x when f(x) = 120 in the equation $y = 5^{x+3} – 5$?

a) x = 3

b) x = 1

c) x = 0

d) x = 1/2

Answer: c

Explanation
Substitute 120 for y in $y = 5^{x+3} – 5$. Next add five to both sides to get $125 = 5^{x+3}$. Express 125 as a power of the base 5. $125 = 5^3$. Since both sides have the same base, their exponents must equal each other. 3 = x + 3, so x = 0. Check this answer: $120 = 5^{0+3} – 5 = 125 - 5 = 120$.

Question 2

Which ordered pair is on the graph of $y = (1/4)^x$?

a) (2, 8)

b) (-2, 16)

c) (1/4, 1/16)

d) (1, 0)

Answer: b

Explanation
When x = -2, $(1/4)^{-2} = \frac{1}{\frac{1}{16}} = 16$

Question 3

Which choice correctly describes the graph of $y = 2^{x+3} – 5$?

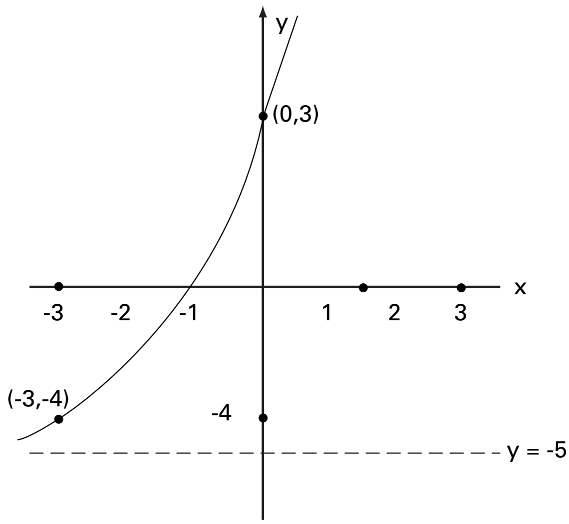
a) An increasing function that approaches y = –5 as x decreases.

b) A decreasing function that approaches y = 5 as x increases.

c) An exponential equation that goes through (0,1).

d) An increasing function that is shifted 3 units right.

Topic 3 – Practice Exam Questions



Answer: a

Explanation
The reference equation is $y = 2^x$. As x decreases you can see that y = -5 is an asymptote. The function itself, however, is an increasing one because as x increases so does y, towards infinity. Use your calculator to see that you get outputs from this equation that are closer and closer to -5 as you let x get closer to -• . The graph of $y = 2^x$ has been shifted 5 units to the right and 5 units down.

Question 4

$A = P \times e^{rt}$ is the formula for the amount of a sum that is compounded continuously at a fixed rate. A sum of $25,000 is invested for 6 years at 7%. What will the balance be after 4.3 years?

a) $38,650

b) $33,432

c) $35,729

d) $33,780

Answer: d

Explanation
We need to find A when P = $25,000, r = .07, and t = 4.3.

Begin by calculating the exponent and then calculate e raised to this number using the e^x key.

$A = \$25,000 \times e^{(.07)(4.3)}$

$A = \$25,000 \times e^{.301}$

A = $25,000 x 1.351209

A = $33,780

Topic 3 – Practice Exam Questions

Question 5

$f(x) = \left(\frac{1}{8}\right)^x$ What is true about this function?

a) increasing, domain (-° , °), range (0, °)

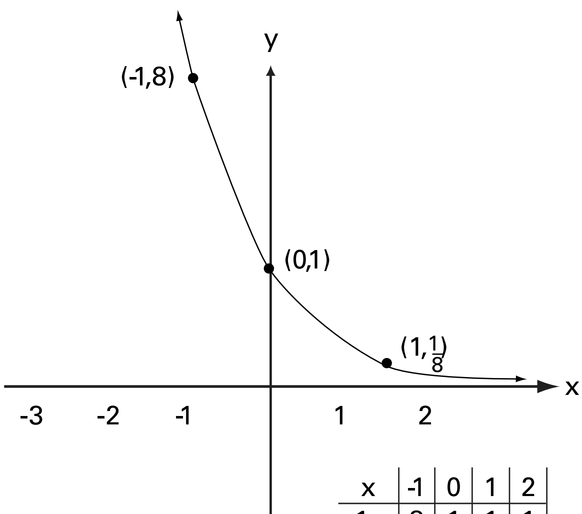
b) decreasing, domain (-° , °), range (0, °)

c) decreasing, domain (-° , °), range (0, °)

d) increasing, domain (-5, °), range (0, °)

Answer: c

Explanation
This exponential equation is in basic form, and there are no shifts involved. Therefore, you know that (0,1) is on the graph. Because (0 < 1/8 < 1) you also know that the function is decreasing as x increases. The positive x axis is an asymptote. The input numbers (domain) for exponential equations in standard form are between (-• , •), and the output numbers (range) are (0,•). You can also plot the graph using a table of convenient values for x.



x	-1	0	1	2
$\left(\frac{1}{8}\right)^x$	8	1	$\frac{1}{8}$	$\frac{1}{64}$

Unit 1 – Equations and Patterns

UNIT 1

Topic 4 – Logarithmic Functions

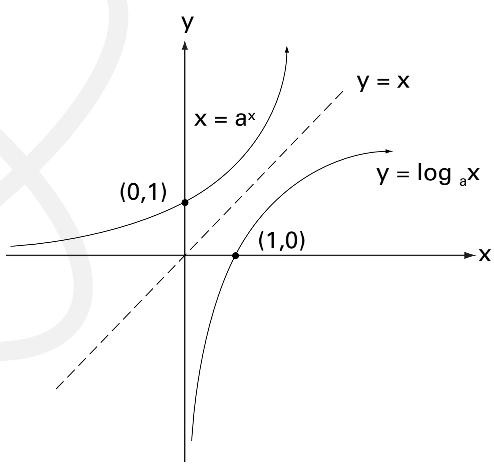
Logarithms Are Exponents

The Logarithmic function undoes what the exponential function does. The range and domain are reversed. The logarithmic function is the inverse of the exponential function and vice versa.

> **Remember: Open intervals do not include their endpoints**
>
> (0, °) means all of the values between, but not including, 0 and ° .
>
> (0, °) = (0 < x < °)

Recall that the domain of the exponential function $y = a^x$ has domain (-° , °) and range (0, °). This is reversed for $y = \log_a x$. **There are no logarithms of negative numbers or of zero because the domain is** (0, °) . Try entering a negative number and taking its log on a calculator- you will get an error message. For example there is no $\log_a 0$ because 0 is not in the domain. The negative y axis is a vertical asymptote. $\log_a x$ approaches -° as x decreases. X gets closer to 0 but never reaches it.

> **Important**
>
> When a logarithm is used in an algebraic expression, for example $\log_a (x-3)$, the values permitted for the variable (x) are limited to those that make the logarithm a positive number. (x-3) must be a positive number to be meaningful in $\log_a (x-3)$. The variable in an expression, however, may be able to take on some negative values.
>
> **Examples**
>
> $\text{Log}_a (x+2)$ is only defined for (-2< x < °) because any values equal to –2 or less are not defined in the function $y = \log_a x$. x can have some negative values, namely those between, but not including, -2 and 0.
>
> $\log_e (x – 1/2)$ is only defined for (1/2 < x < °) because any values equal to 1/2 or less produce a negative number and $y = \log_e x = \ln x$ is not defined for negative numbers.



Topic 4 – Logarithmic Functions

This diagram compares the general form of the exponential function with its inverse, the logarithmic function. You can see that the graphs are reflections of each other with the line y = x as the "mirror" of reflection. You saw how transformations of the exponential function can be graphed and represented in equations in the last topic. A similar discussion for logarithmic equations is given below.

A logarithm is a number used as an exponent

$2^2 = 4$ ´ $\log_2 4 = 2$. This says that the base 2 raised to the second power equals four if and only if the logarithm (i.e. exponent) to the base 2 of 4 equals 2.

$y = \log_a x$ means that $a^y = x$. Be sure to note that y is an exponent. $\log_a x$ is the exponent that the the base, a, is raised to in order to produce x. The subscript in $\log_a$ indicates the base. When a base is omitted the log is understood to be base 10. Logarithms to the base 10 are also called common logarithms. When the base is understood to be e, the symbol ln is used. **Logarithms with the base e are also called natural logarithms**. A scientific calculator will have keys for each function as well as keys that raise e to any power and 10 to any power.

Study the next table to see how a logarithmic form for an equation has an equivalent exponential form and vice versa. ´ means "if and only if".

Logarithmic form	´	**Exponential form**
$4^2 = 16$		$\log_4 16 = 2$
$25^2 = 625$		$\log_{25} 625 = 2$
$.1 = 10^{-1}$		$\log .1 = -1$
$\frac{1}{8} = 2^{-3}$		$\log_2 \frac{1}{8} = -3$
$35 = e^{3.3333}$		$\ln 35 = 3.55534$
$1/2 = e^{-.693147}$		$\ln -.693147 = 1/2$

Topic 4 – Logarithmic Functions

Moving between these forms helps us to evaluate logarithms. Logarithmic equations are solved by a combination of evaluating them and using their properties as explained below.

Examples

1. $\log_{1/2} 8$ is the exponent that 1/2 must be raised with to produce 8. The base is 1/2. We can write an equation in exponential form to find this exponent :

   $\left(\frac{1}{2}\right)^x = 8$

   $(2^{-1})^x = 2^3$

   $(2)^{-x} = 2^3$

   $-x = 3$

   $x = -3$

   Check: $\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$

2. $\log_2 \frac{1}{8} = y$

   $2^y = \frac{1}{8}$

   $2^y = 2^{-3}$

   $y = -3$

   Check: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

3. In (-6)
   We could try $e^x = -6$, but -6 is negative and is not defined in the domain of the natural logarithm function. There is no power of e that will produce -6.

4. In 1
   $e^x = 1$

   $x = 0$ because $e^0 = 1$

5. Find x in $\log_8 4 = x$
   In exponential form this is equivalent to $8^x = 4$

   $2^3 = (2^2)^x$

   $2^3 = 2^{2x}$

   $3 = 2x$

   $x = 2/3$

   check: $\log_8 4 = 2/3$ ´ $4 = 8^{2/3}$

   $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$

Unit 1 – Equations and Patterns

Topic 4 – Logarithmic Functions

Properties of the logarithmic function $f(x) = \log_a x$

1. The graph of $x = \log_a y$ is the reflection of $y = a^x$ across the line $y = x$.
2. The function increases for $a > 1$, and decreases for $0 < a < 1$.
3. The x intercept is (1,0) (the inverse of the exponential function).
4. The y axis is a vertical asymptote.
5. The domain is $(0, \infty)$ (only positive numbers can have logarithms).
6. The range is $(-\infty, \infty)$.
7. Like the exponential function, the logarithmic function is one to one and we can solve logarithmic equations by using this property:

   If $\log_a x_1 = \log_a x_2$, then $x_1 = x_2$ for $a > 0$ and $a \neq 1$.

Operations with logarithms are based on their properties

For a, B, C > 0, with $a \neq 1$, and x any real number:

1. $\log_a (a) = 1$ and $\log_a (1) = 0$
2. $a^{\log_a x} = x$ and $\log_a a^x = x$ and $a^{\log_a B} = B$
3. $\log_a (B^x) = x \log_a B$ and $\log_a a\frac{1}{B} = -\log_a B$
4. $\log_a (BC) = \log_a B + \log_a C$ BUT: $\log_a B \times \log_a C \neq \log_a(BC)$
5. $\log_a \left(\frac{B}{C}\right) = \log_a B - \log_a C$ BUT: $\frac{\log_a B}{\log_a C} \neq \log_a B - \log_a C$

Use these properties to solve logarithmic equations.

Change of Base theorem

$\log_a x = \frac{\ln x}{\ln a}$

or more generally, $\log_a x = \frac{\log_b B}{\log_b a}$ where b is any positive base

Remember: a and x must be positive and $a \neq 1$

This property allows us to express any logarithm in terms of the natural logarithm or any other base logarithm. The reverse is also useful: It allows us to express the ratio of two natural logs or any other same-base logs in terms of a single logarithm with base equal to the denominator.

Examples

$$\frac{\ln 7}{\ln 5} = \log_5 7 = \frac{1.94591}{1.60943} \cong 1.20906$$

Unit 1 – Equations and Patterns

Topic 4 – Practice Questions

Question 1

What is the logarithmic form of $4^3 = 64$?

a) $3 \log_4 = 16$

b) $\log_3 4 = 64$

c) $\log_4 64 = 3$

d) $\log_4 3 = 16$

Answer: c

Explanation
A logarithm is an exponent, and this question asks for the logarithmic expression of the fact that 4 raised to the 3rd power equals 64. The base is four, the exponent is 3, and the result of 4^3 is 64. $\log_4 64 = 3$ is read, " the log of 64 to the base 4 is 3."

Question 2

What is the exponential form of $\log_4 2 = 1/2$?

a) $4^{1/2} = 2$

b) $1/2^2 = 4$

c) $1/4^2 = 2$

d) $\sqrt{2^2} = 2$

Answer: a

Explanation
We need an expression that says 4 raised to the 1/2 power equals 2. The exponent 1/2 means the square root. $(4)^{1/2} = \sqrt{4} = 2$. Notice that choice d is a true statement, but it does not use the base four as required by the question.

Unit 1 – Equations and Patterns

Topic 4 – Practice Questions

Question 3

Solve for x in $\log_x 0.01 = -2$

a) $x = .1$

b) $x = 10$

c) $x = -1/2$

d) $x = 1$

Answer: b

Explanation
Rewrite in exponential form and then solve:

$x^{-2} = 0.01$

$$\frac{1}{x^2} = \frac{1}{10^2}$$

$x^2 = 10^2$

$x = 10$

Question 4

Simplify $\log_3 (\log_4 4)$?

a) 1

b) 1/2

c) 12

d) 0

Answer: d

Explanation
Notice that $\log_4 4 = 1$ because $4^1 = 4$. Now the problem becomes $\log_3 1$, and we know that $3^0 = 1$, therefore $\log_3 1 = 0$.

Topic 4 – Practice Questions

Question 5

Find x in $\log_8 4 = x$

In exponential form this is equivalent to $8^x = 4$.

a) $2^2 = (2^3)^x$

b) $2^2 = 2^{3x}$

c) $2 = 3x$

d) $x = 2/3$

Answer: a

Explanation
To solve this equation first re write it in logarithmic form: $x = \log_3 5$. now use the change of base theorem with a choice of a new base. Here we use common logs to get the answer $\frac{\log 5}{\log 3}$. You can verify this answer on a calculator.
$\frac{\log 5}{\log 3} = 1.4649$, and $3^{1.4649} = 5$. Try the solution with $\frac{\ln 5}{\ln 3}$ and you will get the same answer.

Question 6

Solve this equation: $\log_3 x + \log_3 (x - 2) = 1$

a) $x = 1, x = 0$

b) $x = 3$

c) $x = 3, x = -1$

d) $x = -2, x = 1$

Answer: b

Explanation
We first use the property that the sum of two logs with the same base is equal to the log of the product: $\mathbf{\log_a (BC) = \log_a B + \log_a C}$.
This gives us $\log_3 (x)(x - 2) = 1$

Next change this into exponential form: (notice that the product plays the role of the number that results from raising 3 to the first power)

$(x)(x-2) = 3^1$

$x^2 - 2x - 3 = 0$

$(x + 1)(x - 3) = 0$

$x = -1$

$x = 3$

Topic 4 – Practice Questions

This quadratic equation has two roots, and we need to check whether they produce 1 when substituted into $\log_3 x + \log_3 (x - 2) = 1$. We reject –1 because a log cannot be found for a negative number. Only x = 3 is a solution.

Question 7

A study of how fast the flu virus can spread in a population when two people with the flu arrive, gives the equation:

$T = -1.43 \ln \frac{10{,}000 - n}{4998n}$ where T is the time in days required to infect n people.

If two people arrive at the Snap Lake diamond mine in the Northwest Territories with the flu, and there are 330 people in the camp, how long will it take to infect everyone in camp?

a) about two weeks

b) about two days

c) about 5 days

d) about 7 days

Answer: d

Explanation

We need to know how long it will take for n = 330 people to be infected. The calculations require the use of the lnx key on a calculator.

$$T = -1.43 \ln \left(\frac{10{,}000 - 330}{(4998)(330)} \right)$$

$$T = -1.43 \ln (.00586295)$$

$$T = -1.43(-5.139)$$

$$T = 7.35$$

The entire population will be infected in a little more than one week.

Question 8

Use the stepwise formula for compound interest $A = P\left(1+\frac{r}{n}\right)^{nt}$, to find how long it will take for $3000 to triple in value if it is invested at 7% and compounded daily.

a) 10.29 years

b) 15.69 years

c) 12.33 years

d) 14. 78 years

Answer: b

Explanation
We need to find t when n = 365 days (daily compounding), r = .07, P = $3000, and A = $9000, which is 3 x $3000. The change of base formula will help with the solution.

$$9000 = 3000\left(1+\frac{.07}{365}\right)^{365t}$$

$3 = (1+ .00019178)^{365t}$

Now use the definition of logarithm to see that

$365t = \log_{1.00019178}(3)$

This says that the exponent to which 1.00019178 must be raised to equal 3 is 365t.

Next use the base change formula to put the right side of the equation into the form, then $\frac{\ln 3}{\ln 1.00019178}$

$$365t = \frac{\ln 3}{\ln(1.00019178)} = 5729.05$$

$$t = \frac{5729}{365} = 15.69$$

This means that $3000 invested at 7% with daily compounding, will triple in 15.69 years.

Topic 4 – Practice Questions

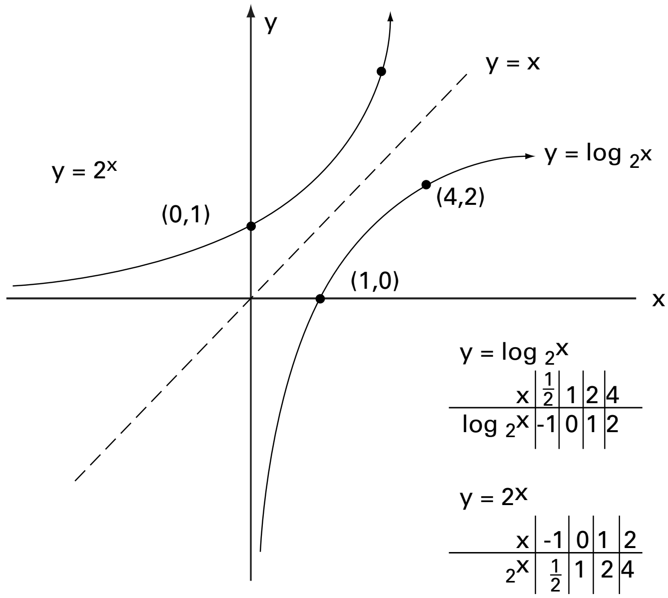
Graphing Logarithmic Functions

Just as with exponential functions, there are two basic shapes for the reference function $y = \log_a X$ depending on whether the base, a, is between 0 and 1, or greater than 1. Vertical and horizontal shifts that transform the graph are analyzed in a similar way to how they were handled for exponential functions.

Examples

1. Graph $y = \log_2 x$

   This graph will be the reflection across y = x of $y = 2^x$



$y = \log_2 x$

x	$\frac{1}{2}$	1	2	4
$\log_2 x$	-1	0	1	2

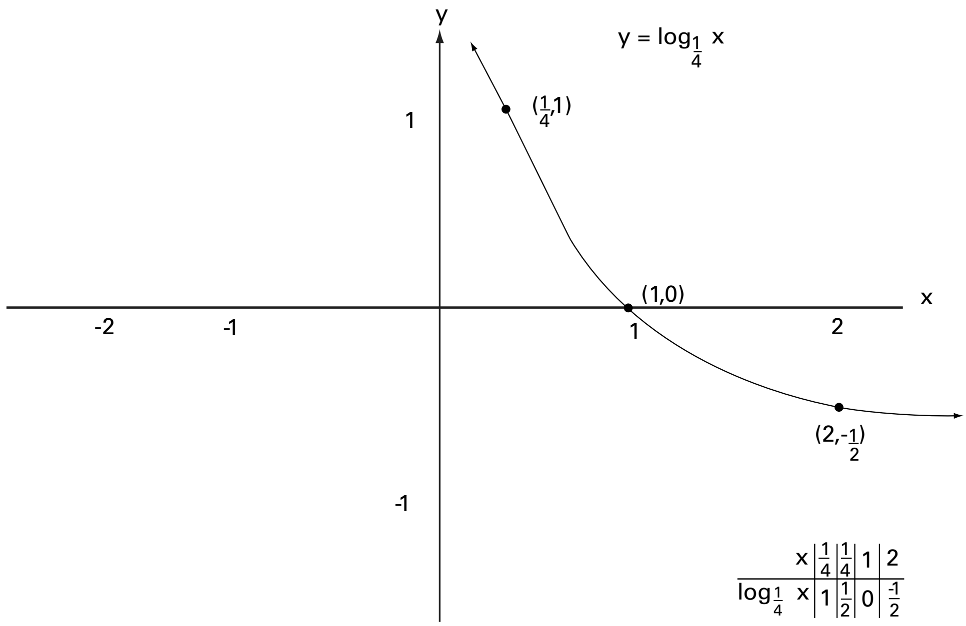
$y = 2^x$

x	-1	0	1	2
2^x	$\frac{1}{2}$	1	2	4

You can see from the graph and the tables that these are inverse functions. The range and domains change places. Notice the properties discussed earlier. The domain of $y = \log_2 x$ is (0, °) because $\log^2 x$ approaches 0 as x gets smaller, but it never reaches it. 0 is not in the domain. (The range goes from -° , °). This situation is reversed for the exponential function.

Topic 4 – Practice Questions

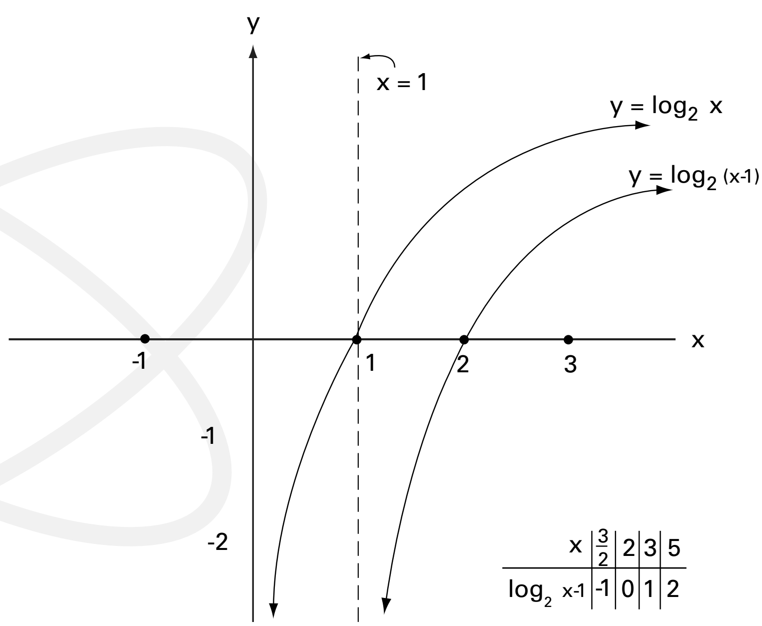
2. Graph $y = \log_{1/4} x$



x	$\frac{1}{4}$	$\frac{1}{4}$	1	2
$\log_{\frac{1}{4}} x$	1	$\frac{1}{2}$	0	$\frac{-1}{2}$

This is a decreasing function because $\log_{1/4} x$ approaches -° as x increases. The y axis is a vertical asymptote and the range of the function is ° . The domain is (0, °).

3. Sketch the graph of $y = \log_2 (x - 1)$. This is a transformation of the reference function $y = \log_2 x$. The graph of $y = \log_2 x$ is moved one unit to the right because –1 is added to x. If 1 were added, the move would be one unit left. This changes the domain from (0, °) to (1, °). The vertical asymptote changes from the y axis to the line x = 1.



x	$\frac{3}{2}$	2	3	5
$\log_2 x\text{-}1$	-1	0	1	2

Unit 1 – Equations and Patterns

Topic 5 – Quadratic Equations (Supplementary Topic)

Background

A quadratic equation has a variable that is squared. To solve these equations we must find out the value of this variable, usually represented by the letter x. Quadratic equations will have two solutions for the variable that is squared because there are two square roots of any positive number. -2 squared, and 2 squared, both have four as the answer. In general, the square root of any positive number is really two numbers, the positive and negative root. For example, the square root of 16 is both 4 and -4, the square root of 25 is both 5 and -5. Quadratic equations will have two answers, one positive and one negative.

In order to solve a quadratic we put them into **standard form**. The standard form is:

$$ax^2 + bx + c$$

Where a, b and c are numbers, and a is not equal to 0.

Solving quadratic equations requires that we put them into standard form and then plug the values of a, b, and c, into the solution formula.

The two solutions for x are given by this formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Memorize this solution formula and use it to solve the problems that follow. It's a good idea to write this formula on a piece of paper as soon as you begin the exam so you have it when needed. Sometimes people work on the calculation under the square root sign and forget to divide the numerator by 2a. Be sure to work step by step and keep the formula in view.

Sample Problems

1. Solve for x in this equation: $3x^2 = -7x + 5$

   This is a quadratic because the variable in it is squared. We need to put it into standard form by rearranging the terms. Begin by moving the terms on the right to the left side of the equal sign. You will then have:

   $3x^2 + 7x - 5 = 0$

   This is in standard form. Now you can use the quadratic formula. Identify the values for a, b, and c.

   a = the coefficient of x^2 = 3

   b = the coefficient of x = 7

   c = the number without a variable coefficient = -5

Substitute into the formula to get the expressions that we will simplify and evaluate:

$$x = \frac{-7 + \sqrt{7^2 - (4)(3)(-5)}}{(2)(3)} = \frac{-7 + \sqrt{109}}{6} = \frac{3.44}{6} = .5733$$

$$x = \frac{-7 - \sqrt{7^2 - (4)(3)(-5)}}{(2)(3)} = \frac{-7 - \sqrt{109}}{6} = -\frac{17.44}{6} = -2.90667$$

These expressions have been evaluated, but the exam may give choices that stop short of evaluating the expression.

2. Find the solution for x in the equation $(x - 2)(x + 3) = 5$
This might not look like a quadratic, because there is no x^2 term. However, when you distribute the terms by multiplying you will get a term with x^2 in it. Begin by multiplying the expressions and put the result into standard form.

$(x - 2)(x + 3) = 5$

$x^2 + 3x - 2x - 6 = 5$ (multiply all terms)

$x^2 + x - 6 = 5$ (simplify)

$x^2 + x - 11 = 0$ (subtract 5 from each side)

The equation is now in standard form. Identify a, b, and c, and use the quadratic formula that you have memorized.

$a = 1$

$b = 1$

$c = -11$

$$x = \frac{-1 \pm \sqrt{1 - (4)(-11)}}{2} = \frac{-1 \pm \sqrt{45}}{2}$$

These two roots, or answers, are also called the solution set for the quadratic equation. Simplify the square root of 45 to $3\sqrt{5}$ for a final answer of:

$$\frac{-1 \pm 3\sqrt{5}}{2}$$

In words this solution set is, "minus one plus or minus three times the square root of five all divided by two."

3. Solve this equation: $x^2 + 5x + 4 = 0$

   This is a quadratic equation already in standard form.

   a = 1

   b = 5

   c = 4

   $$\frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2}$$

   x = -1 and x = -4

   Check:

   x = -1

   $x^2 + 5x + 4 = 0$

   1 – 5 + 4 = 0

   0 = 0

   x = -4

   $x^2 + 5x + 4 = 0$

   16 – 20 + 4 = 0

   0 = 0

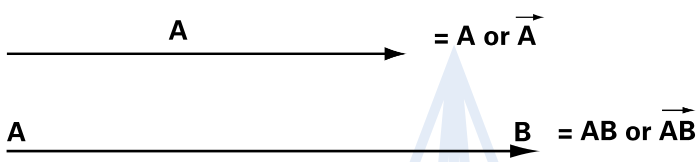
Unit 2

Vectors

Topic 1 – Vectors and Scalars

Vectors are quantities that have both magnitude and direction.[22] They are useful in work that involves force or changes in an object's position (displacement). We use vectors to predict where a moving object will wind up and what the overall or net effect of a combination of forces acting on an object will be.

Vectors can be shown by directed line segments. Bolded capital letters are used to represent vectors and distinguish them from scalars. An arrow can also be placed over a letter or pair of letters to indicate a vector. In a scaled diagram, an arrow head points in the **direction** of the vector and the length of the segment indicates the **magnitude**. The symbol for absolute value is used to refer to the magnitude of a vector. $|\vec{V}|$ = the magnitude of **V**. In 30 mph west, 30 mph is the magnitude, and west is the direction. If **V** is used to represent this vector, then $|\vec{V}|$= 30mph.[23] A vector can also be represented by a pair of letters that label the tail and tip of a vector arrow.



22 In this unit we will study examples in two dimensions only. Operations on vectors in three dimensional space extend the principles of vector operations to directed line segments in a three dimensional coordinate system. The addition of a third dimension, the z axis in a coordinate system, requires ordered triples (x, y, z) to identify a point in three dimensional space. Angles that determine direction will relate a vector in three space to three angles corresponding to the vector and each axis.

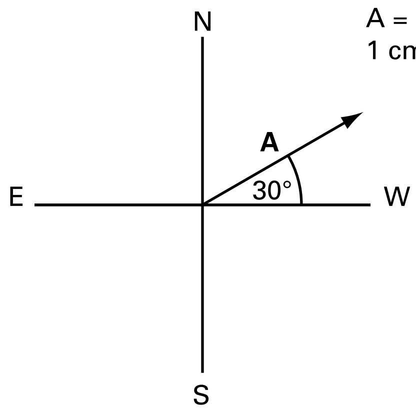
23 In this curriculum both ways of representing vectors will be used, bold type capital letters, and capital letters with an arrow over them.

Unit 2 – Vectors

Topic 1 – Vectors and Scalars

Unlike **scalar quantities,** vectors indicate the direction that a number is applied to. 3 miles west is a vector, but 3 miles is a scalar. 3 lbs. is a scalar, but 3lbs. of force directed at an angle 30 degrees north of east is a vector. Force, velocity and acceleration are vectors. Scalar quantities may have units of measurement but do not have direction. 30 mph is a scalar, as is the number 30 by itself, but 30mph blowing 30° north of west across the east arm of Great Slave Lake is a vector. Angles corresponding to compass directions are often used to indicate the direction of a vector.

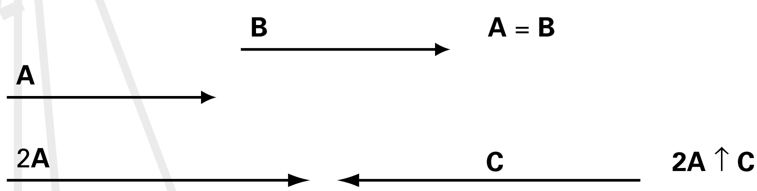
A = 30 mph 30° North of West
1 cm = 10 mph



The length of a vector in a diagram can be scaled to represent its magnitude. In the next diagram, a vector of 5 cm represents a wind with a magnitude of 50 mph. It is not always necessary to name the endpoints of a vector, but every vector will have two endpoints. The length of a vector in a scale drawing is determined by the magnitude of the vector. Its terminal end is indicated by an arrow tip. The usefulness of vectors includes their ability to represent a directed quantity on a point or object as well as on any point or object in a field. For example, a 30mph south wind can be represented by any vector with magnitude 30 that points south. All of the parallel vectors to it with the same magnitude will be equal. The tail and tip do not have to connect with specific objects.

Properties of Vectors

1. Two vectors are equal if they have the same magnitude and direction – even if they are in different places. This implies that equal vectors will be either parallel or lie on parts of the same line (i.e. be co-linear).



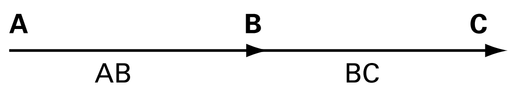
Unit 2 – Vectors

Topic 1 – Vectors and Scalars

A = B because they have both the same length (magnitude) and direction. 2**A** has twice the magnitude of **A** and points in the same direction. 2**A** is an example of a scalar multiplying a vector. **A scalar coefficient does not change the direction of a vector**. Notice that 2**A** does not equal **C** even though they have the same length. **C** has the reverse direction to 2**A**. **A** is not equal to **C** because they have different directions even though their lengths (i.e. magnitudes) are equal.

> **Scalar Multiplication**
>
> A scalar k times a vector **V** creates a new vector k**V** with magnitude = |k| x the magnitude of **V**. For k greater than 0, k**V** will have the same direction as **V**. For k less than 0, k**V** will have the opposite direction to **V**. If k = 0, then k**V = 0**.

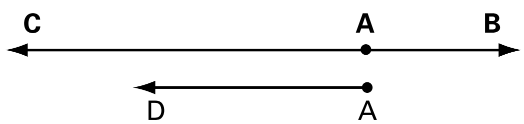
2. The **zero vector** (**0**) has no magnitude and no direction.



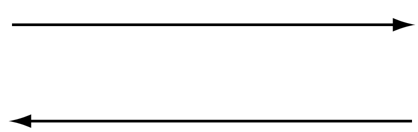
3. Vectors that overlap are **collinear**.

   AB is collinear with **AC** and **BC** is collinear with **AC**.

5. **Opposite vectors** point in opposite directions. If they act on the same point, i.e. have their tails at the same point, the resultant vector will be their difference in magnitude in the direction of the larger vector. An opposite can be produced for every vector with the same magnitude and opposite direction. When combined with its opposite, the result will be the zero vector.



AC is an opposite vector to **AB**. They are also collinear, and they have different magnitudes. The result will be a vector **AD** with its tail at **A** and its tip at a point D in the direction of **C** with a magnitude equal to that of $|\overrightarrow{CA}| - |\overrightarrow{AB}|$. This is an example of a vector sum, or resultant.



In this diagram **A** is opposite to **B**. They have the same magnitude and are also parallel but they have opposite directions. They will combine to give the zero vector. **A + B = 0**. Also notice that **A = -B** and **B = -A**. The negative sign means the direction is reversed. Because **B** and **A** are anti-parallel, they become equal when the direction of one of them is reversed.[24]

24 You may want to review operations on signed numbers in Math foundations. Travel on a number line with signed numbers is a form of vector addition because + and – are used to indicate direction.

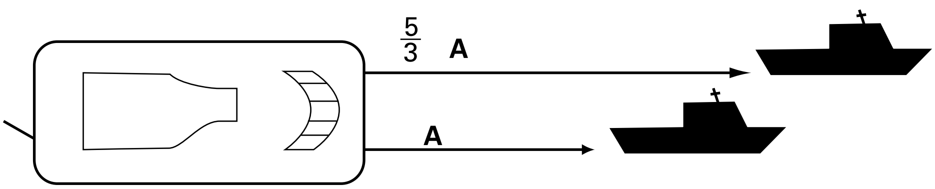
Unit 2 – Vectors

Topic 1 – Vectors and Scalars

Multiplying and Adding Vectors With Scalars

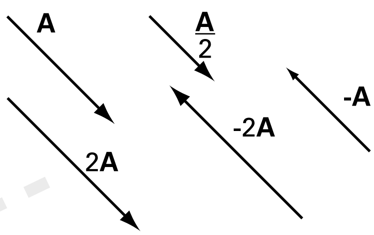
Multiplication is a form of repeated addition. Two boats pulling a fishing factory ship into Yellowknife's back bay illustrate the idea of combining parallel vectors that act on the same object. The force of the stronger boat can be expressed as a scalar multiple of the weaker boat's force. For example, if one boat pulled with 5000 lbs of force and the other boat travelled parallel to the first and pulled with 3000 lbs, the combined pull on the ship would be 8000 pounds in the direction of the tow.
If A = 1000 pounds = 1 cm , we would have this picture:

Scale:
1 cm = 3000 lbs



5/3 **A** expresses the magnitude of the stronger tug's force as a multiple of the weaker tug's force. The lengths of the vector arrows show the relative magnitudes of the force exerted by the two tugs. The vectors show multiplication of a vector by a scalar that relates their magnitudes. In this diagram the scale sets **A** = 3000 lbs = 3cm. You can see the addition of the two vectors as the sum of their magnitudes in the same direction.

If the two vectors were not parallel, their magnitudes could not be added directly. If **A** is chosen as the unit vector, then other parallel vectors in the direction of **A** are expressed as scalar multiples of **A**. Parallel vectors in the opposite direction are expressed with a negative sign.



In this diagram you can see a group of vectors with opposite direction and different magnitudes. **A**, **A**/2, and 2**A** are opposite in direction to -**A** and -2**A**.

A has been given a positive value. Some combinations of these include

2**A** + (-2**A**) = **0**

-**A** + **A**/2 = -**A**/2

A + 2**A** + **A**/2 = 7**A**/2

-2**A** + **A**/2 = -3**A**/2

-**A** + (-2**A**) = -3A

Convince yourself that these algebraic combinations are true by the overlapping vector method of diagramming. Take the tail of each vector arrow in the last diagram, and place it on the tip of the parallel vector arrow it is being combined with. The result will show the magnitude of the combination in the direction of the longer (ie. larger magnitude) vector.

When vectors with the same direction are added their length is the sum of the lengths, when vectors with opposite directions (signs) are added, the resulting length will be the difference between the longer arrow and the shorter one- pointing in the direction of the longer arrow.

Unit 2 – Vectors

Topic 1 – Practice Exam Questions

Question 1

Which quantity is a vector?

a) Two to one odds

b) 30°

c) 32 feet per second 20° from the horizontal

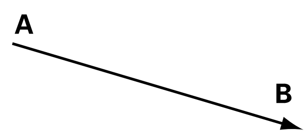
d) 50 pounds per square inch

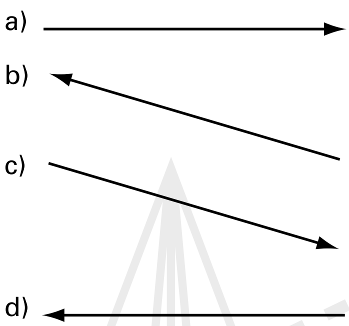
Answer: c

Explanation
Only c combines a quantity with a direction. Choice a is a ratio comparing two scalars. The other choices are scalars. Choice d describes units of area, but has no direction, and choice b describes the magnitude of an angle with no information about its direction.

Question 2

Which vector is equal to **AB**?





Answer c

Explanation
We need a vector that has the same length and points in the same direction. The vector in choice c is parallel to **AB**. Only choice c does this. The other vectors have the same magnitude as **AB**, but they have different directions.

Topic 1 – Practice Exam Questions

Question 3

One man pulls on a crate with 50 lbs. of force to the west, and another pulls with 30 lbs of force to the east. Which vector will represent the result?

a) The zero vector

b) A vector pointing east with a magnitude of 30 pounds

c) A vector point west with a magnitude of 20 pounds

d) A vector pointing west with a magnitude of 20 pounds

Answer: d

Explanation
A sketch will show two opposed vectors with their tails on the crate. 50 lbs west minus 30 pounds east leaves 20 pounds west. Because the vectors are opposites, the larger magnitude will determine the resulting direction.

Question 4

A vector with magnitude 10 points North . An opposite vector has a magnitude of 20. What will be the magnitude and direction of the resultant?

a) a vector with magnitude 10 pointing south

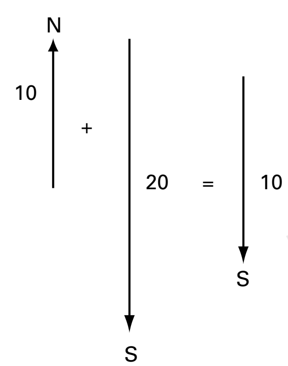
b) a vector with magnitude 20 pointing north

c) a vector with magnitude 10 pointing north

d) a vector with magnitude 0 and no direction

Answer: a

Explanation
Because the vectors are parallel their magnitudes can be combined and their directions indicated by signs. Algebraically, if south is taken to be the negative of north then the resultant equals -20 + 10 = -10 = 10 South. If North is taken to be the negative of south then 20 –10 = 10 = 10 south.



Unit 2 – Vectors

Topic 1 – Practice Exam Questions

Notice that the addition of parallel vectors does not require them to be collinear. Notice also that opposite vectors can be represented algebraically by negating the magnitude of one of them and then by combining the magnitudes. The direction of the result will be that of the component vector with the greatest absolute magnitude. You can also see that the tip to tail method will produce overlapping vectors and give the same answer.

Question 5

What is the sum of **-A**, **B**, and **-A**/2

a) **B + A**

b) **B – A**/2

c) **B** + 2**A**

d) **B** – 3 (**A**/2)

Answer: d

Explanation

Opposite signs indicate opposite directions on a vector. **-A** combines with **-A**/2 to give $-\frac{3\vec{A}}{2}$. B cannot be simplified or combined , so it is left as a vector to combine with $-\frac{3\vec{A}}{2}$

Unit 2 – Vectors

Topic 2 – Vector Operations

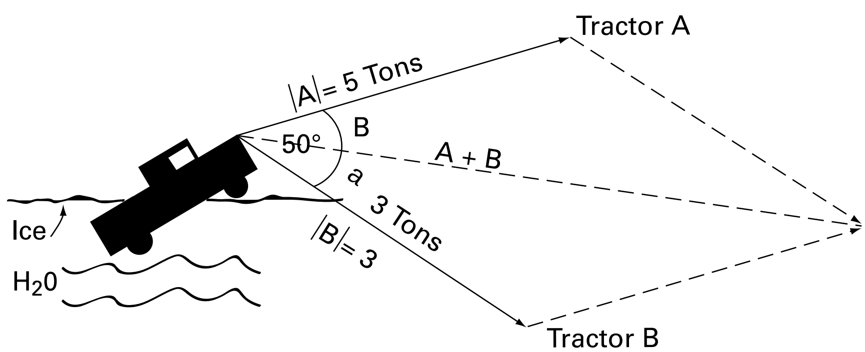
Vectors can be combined by operations that take into account their direction as well as their magnitude.

Resultant Vectors

When vectors are not parallel and/or collinear, the net effect (i.e. the resultant) can be calculated by placing the **component vectors** tip to tail and using a triangle method or a parallelogram method to take into account the difference in direction between them. **The resultant is equal to the combined effect of the component vectors**. Displacement (i.e. net or final change of position) is a closely related idea to that of a resultant.[xx]

Examples

1. Consider two tractors pulling a truck to shore that fell through the ice at an angle of 50° from each other. We cannot add the magnitudes (3 tons, and 5 tons), to find the total force being exerted on the truck because there is an angle between the forces. The combined force will be less than 8 tons. In order for the combined force to be 8 tons the tractors would have to be parallel and pull in the same direction. If this was the case, they could be added using the methods in topic 1 above. The component vectors in this example, however, are neither parallel nor anti-parallel, and therefore we cannot combine their magnitudes directly.

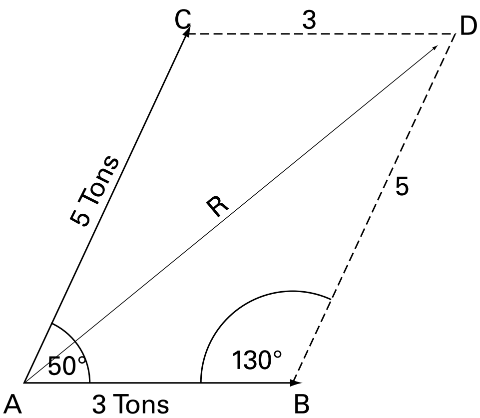


The force that acts on the diagonal **AD** will be the combined force of **AB** and **AC**. **AD** is the sum of **AB + AC**, and these are called the **components** of the sum. **AD** will have the same effect (i.e. result) as the combined effect of **AB** and **AC**. **AB + AC = AD.**

Vector addition is done by a **parallelogram method**. Complete a parallelogram by placing tip to tail vectors that use the fact that parallelograms have equal opposite sides. Note that the diagonal of a parallelogram is not an angle bisector, and that the adjacent angles are supplementary. We solve for the angles and length (magnitude) that define the diagonal (i.e. the resultant vector) by using the trigonometric identities of the law of sines and the law of cosines.

25 See Science Special Topics for a discussion of displacement calculations using vectors

Topic 2 – Vector Operations



To find **R** solve the triangle involved with the **law of cosines**. This formula relates the square of a side in an oblique triangle to the other two sides and their included angle. In this case the included angle = 180° – 50° = 130°, and the included sides are 3 and 5.

> **Law of Cosines**
>
> In an oblique triangle the square of each side equals the sum of the squares of the other two sides minus two times the product of the other two sides and their included angle.
>
> $c^2 = b^2 + a^2 - 2ab \cos g$

Note that g (gamma) is the angle opposite side c. In our problem **R** = c. g is the angle bounded by sides a and b. In our problem we set a=3, b= 5, g = 130°.

$R^2 = 9 + 25 - 2(5)(3) \cos 130°$

$R^2 = 34 - (30)(-.6427)$

$R^2 = \sqrt{53.28}$

$R = 7.3$

The combined force of two tractors will equal 7.3 tons. This is the magnitude of **R**.

We still don't know the direction for **R** however. To do this, we use **the law of sines** to find the angle between the resultant R and one of the component vectors. Either angle will indicate direction and serve our purpose. In this example we chose to find angle DAB.

Unit 2 – Vectors

Topic 2 – Vector Operations

The Law of Sines

The ratio between the sine of an angle and the side opposite the angle is the same for each angle in any triangle.

$$\frac{\sin a}{a} = \frac{\sin b}{b} = \frac{\sin g}{c}$$

In our problem we will set angle DAB = a.

$$\frac{\sin a}{5} = \frac{\sin 130°}{7.3}$$

$$\sin a = \frac{5\sin 130°}{7.3}$$

$$\sin a = \frac{5(.7660)}{7.3} = 5.25$$

$$a = \sin^{-1}.525$$

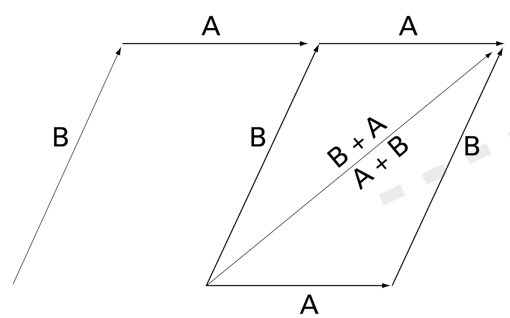
$$= 31.66$$

148.34° is also a solution for a = sin-1.225, (i.e. 180° – 31.66°), but a must be less than 50° in our problem so we choose the solution that fits. We can now give a complete description of **R** as the vector with a magnitude of 7.3 tons acting 31.66° counter-clockwise from the vector **AB**. We can also find the angle CAD = 50° – 31.66° = 18.33°.

It would be just as correct to say that **R** has a direction 18.33° clockwise from **AC**. If compass directions were given in the problem these could be added to the description of the direction.

What you need to know about the parallelogram method

To find the sum (resultant) of any two vectors A + B, place B so its tip touches the tip of A (do not change the magnitude or direction of B when you do this). The resultant will be the new vector that has the same tail as B, and that ends with the same tip as A. Notice that it doesn't matter which vector is chosen to be A or B.

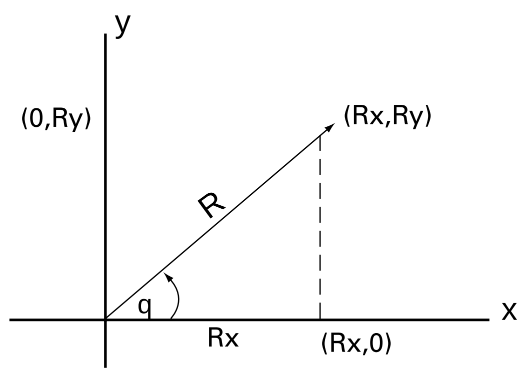


Unit 2 – Vectors

Topic 2 – Vector Operations

Component Methods

The x, y coordinate system can be used to describe vectors. A vector can be represented by a radius vector whose tip touches a point on the circumference of a circle. This is known as the **position vector**. When vectors are added the tip and tail of each will have an (x,y) coordinate. Recall from Unit one the discussion of the radius vector on the unit circle. Everything you know about trigonometric functions can be applied to analyzing vectors. The angle of rotation will determine a vector's direction, and the x and y coordinates of the tip will allow us to find the magnitude of the vector by using Pythagoras' theorem.



The vector R has an x component = R_x, and a y component = R_y We use the notation ·a, bÒ to represent these components of a vector. The direction of R is given by q and the magnitude of R is given by the Pythagorean relationship:

$$|\vec{R}| = \sqrt{R_x^2 + R_y^2}$$

Also $\tan q = \frac{R_y}{R_x}$, $q = \tan^{-1} \frac{R_y}{R_x}$

$\sin q = \frac{R_y}{R}$

and

$\cos q = \frac{R_x}{R}$

This gives the useful equations $R_x = R \cos q$, $R_y = R \sin q$

The components of **R** can be described in several ways:

$\vec{R} = (R_x, R_y)$ = ·a, bÒ = (Rcosq, Rsinq)

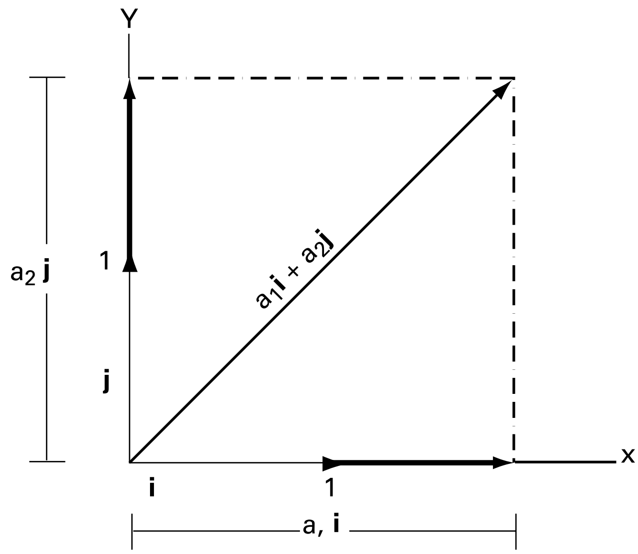
Unit 2 – Vectors

Topic 2 – Vector Operations

With these relationships we can use the components of a vector to do addition and solve related problems. By putting vectors into their component form it is easier to add them.

Unit Vectors

The **unit vector** is the vector with magnitude of one in a measurement system. For example, in the diagram given earlier of wind across Great Slave Lake, 1 cm = 10 mph. A vector with magnitude of 3cm will represent 30 mph and point in the direction of the wind. A standard form for the unit vector uses this idea. In the x, y coordinate system, **i** = ·1, 0Òand j = ·0, 1ÒThese vectors describe vectors one unit long on the x and y axes.



Each has a magnitude of one. Any vector in the form of its components (x,y) can be expressed as a combination of these vectors in the x,y plane, and in the x, y, z space for three dimensions by adding a_3**k** = ·0, 0, 1ÒIn the x, y coordinate system a linear combination of the unit vectors can be found for any vector. It will have the form:

$·a_1, a_2Ò= a_1 ·1, 0Ò+ a_2 ·0, 1Ò= a_1\overset{Æ}{i} + a_2\overset{Æ}{j}$

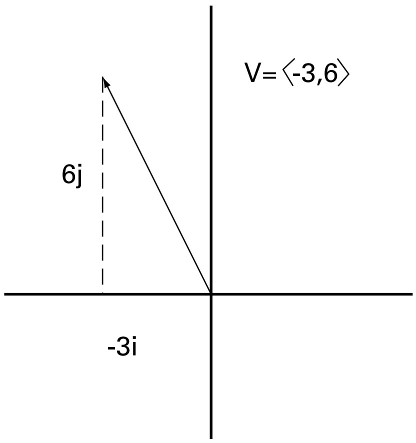
Notice that vectors of different lengths will change the angle of the resultant in the diagram. Multiples of unit vectors can be used to express any vector as the sum of its x and y components. The result of adding the x and y components this way will allow us to express any vector in component form as a linear combination of the unit vectors. For example V = ·-3, 6Òis equivalent to –3i +6j

Unit 2 – Vectors

Topic 2 – Vector Operations

It can be helpful to interpret vector addition as a description of the legs in a journey.

In this example, we travel first 3 units left from the origin on the x axis (-3**i**), and then 6 units up on the y axis (+6**j**). The resultant vector (i.e. the position vector) is the arrow connecting the origin to the point (-3,6). You can use trigonometry to find the magnitude of the resultant vector and the angle that defines its direction. This vector is the result of adding the component vectors.



In this example, the magnitude of the vector is:

$|\vec{V}| = \sqrt{36 + 9} = 3\sqrt{5}$

The direction is given by 180- arctan (–2), because the tangent of the angle between the y axis and the position vector is –2. The angle of rotation from the positive x axis is 116.5°. Check this on your calculator.

The reverse is also true: you can find the components of a position vector as the combination of multiples of **i** and **j**.

What you need to know – Vector combination using components

Examples

Let **T =** ·-3, 2Ò, **U** = ·4, -2Ò

1. **T – U =** ·-3, 2Ò– ·4, -2Ò= ·-7, 4Ò
2. **-5U =** 5·4, -2Ò= ·-20, 10Ò
3. 2**T** + 3**U** = 2·-3, 2Ò+ 3·4, -2Ò= ·-6, 4Ò+ ·12, -6Ò= ·6, -2Ò

Unit 2 – Vectors

UNIT 2

Topic 2 – Vector Operations

The Dot Product

When two vectors are multiplied in component form, their product is defined as the sum of the products of the corresponding components. The law of cosines can be used to show that two vectors are perpendicular if and only if their dot product = 0.

Æ Æ
A B = $a_1b_1 + a_2b_2$

Example

Let **T** = ·-3, 2Ò **U** = ·4, -2Ò

then:

Æ Æ
T U = (-3)(4) + (2)(-2) = (-12) + (-4) = -16

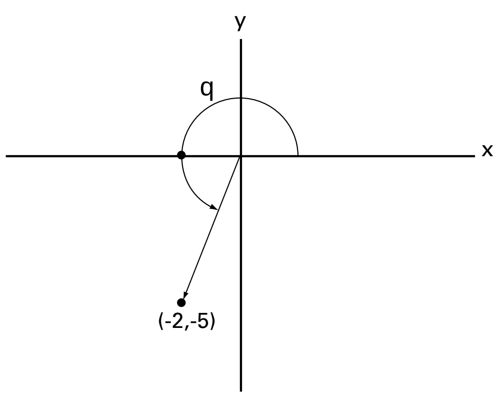
Vector Addition and Polygons

When we add more than two vectors the tail to tip method will allow us to find the resultant in a scale drawing. We can connect the vectors we want to add in several ways by putting tips to tails without changing magnitudes or directions. The resultant will be the same in each case. Connect the tail of the first vector to the tip of the last and that will be the resultant. Experiment with some scale drawings to see how this works.

Examples

1. What is the magnitude and direction of the vector **V** = ·-2, -5Ò?

   We are given the x and y components of V. This allows us to sketch the position vector for V



   Find the magnitude of **V** first so we can use this to calculate sin q and then find q.

   |VÆ| = $\sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$

$$\sin q = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{-5}{\sqrt{29}}$$

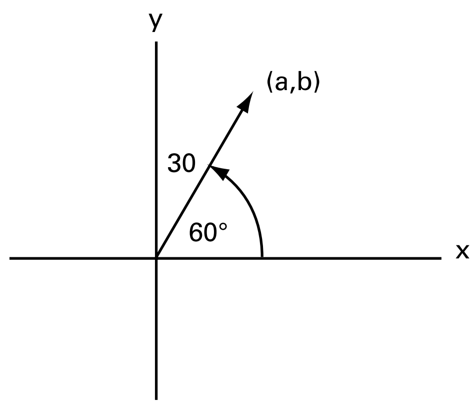
$$\sin^{-1} = \left(\frac{-5}{\sqrt{29}}\right) = -68.2°$$

This is one value for sin q, but we can see from the position of V that we need the other solution on the unit circle, 180° – (-68.2°) = 248.2° We now have a complete description of V = $\sqrt{29}$, 248.2°, which is about 68 degrees south of east.

2. What is the component form for **V** if |V| = 30 mph, and has a direction angle of 60°?

   First make a sketch showing what we are given. The position vector for **V** will be in the first quadrant. We need to find the coordinates of the terminal point.

   We can use the fact that a = |V| cos q, and b = |V| sin q



$a = 30\cos 60°$

$a = 30 \times \frac{1}{2} = 15$

$b = 30\sin 60°$

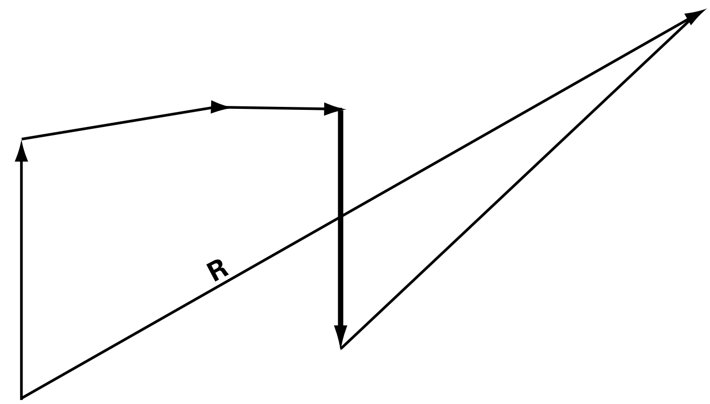
$b = 30\left(\frac{\sqrt{3}}{2}\right) = 15\sqrt{3}$

The component form of V = $\langle 15, 15\sqrt{3}\rangle$

Check: $(15)^2 + (15\sqrt{3})^2 = 900$

$225 + 675 = 900$

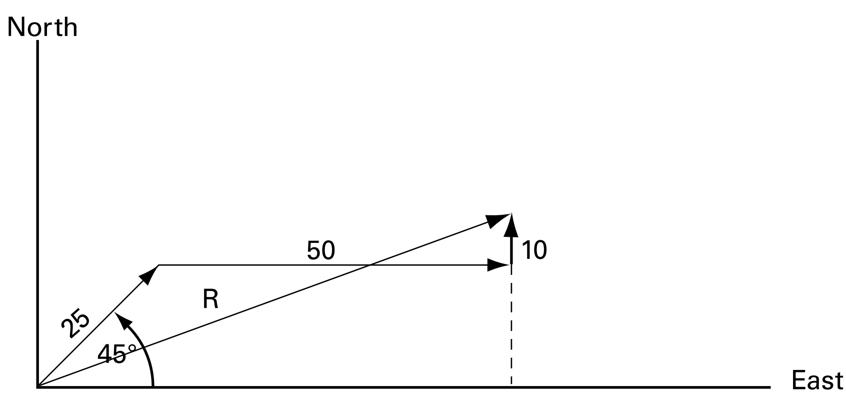
3. What is the sum of the following vectors used to describe a cross country ski route. The scale is 1 cm = .5 km.



The resultant (R) measures 10.5 cm which represents 5.25 km. The sum of the vectors that describe this trip is not their total magnitude, but the magnitude and direction of the resultant vector that connects the start of the trip to its conclusion.

4. A pilot flies 25 km 45° north of east, then changes course and flies 50 km due east, and then 10 km north. How far is the plane from where it started at this point, and in what direction is it flying from the starting point?

   This problem describes three vectors, or displacements. Sketch the information given in the problem and label each vector A, B, and C. We will then find the sum of the vectors which will give us the net displacement and the direction of the resultant.



We can now calculate the components of each vector and solve for the x and y components of the resultant R. Notice that the x component of R equals the sum of the x components of the first two displacements. The y component of R equals the sum of the y components of C and A.

Here are the values we need:

$\vec{A}_x$ = 25cos45° = 17.68

$\vec{A}_y$ = 25cos45° = 17.68

$\vec{B}_x$ = 50 + 17.68 = 67.68

$\vec{B}_y$ = 0 + 17.68 = 17.68

$\vec{C}_x$ = 0 + 67.68 = 67.68

$\vec{C}_y$ = 100 + 17.68 = 27.68

The x component of the resultant R is equal to $A_x + B_x$ = 85.36

The y component of R = $A_y + C_y$ = 45.36

We can find the length of R by using the Pythagorean theorem now that we have the lengths of the right triangle with R as the hypotenuse:

$$|R| = \sqrt{C_x^2 + C_y^2} = \sqrt{(76.68)^2 + (27.68)^2 = } 73.1$$

To find q and describe the direction the plane is flying in from the starting point we can use the arctangent function now that we have the information that gives

$$\tan q = \frac{C_y}{C_x} = \frac{27.68}{67.68} = .40898$$

$$q = \tan^{-1}(.40898) = 22.24°$$

The plane is at a point 73.1 km 22.24° north of east.

Unit 3

Statistics and Probability

Topic 1 – Data Collection and Display

Data is gathered in order to answer questions. For example, a community survey might be conducted to find out how many people own their own homes so that property taxes can be estimated. A questionnaire might be developed to find out how many families would make use of a public daycare facility in the community in order to establish the demand for an investment in daycare.

The results could be listed in a table and a graph could be produced using the methods discussed earlier in this curriculum. In Module Two, bar graphs and line graphs were reviewed. In unit one, topic one, of this module examples were given of a line of best fit using a scatter plot. In these examples tables were used to organize data so that graphs could display the data in ways that allowed us to answer a variety of questions. In particular, we could identify patterns and functional relationships. An equation in the form f(x) = x can then be used to describe the functional pattern.

We continue our review of data collection and analysis with the ideas of a **representative sample** from a population that will allow us to make predictions about the whole population. This is the work of **statistics**. In order for a sample to provide a reliable basis for predictions (aka inferences), it will be representative. For example, if we wanted to know the average wage of working women who have been in the workforce for more than 5 years, our sample would exclude all persons (men and women) who do not fit these criteria.

The people who fit our criteria make up the population we are interested in. In statistics, the population we are interested in is called "the population". The next step would be to find a way to randomly sample the group we are interested in. For example, we might call every third person in a list after ensuring that the list has been "shuffled", or we might look at a cross-section of our population- for example three randomly chosen grade 12 students from every class in every high school in the NT.

Topic 1 – Data Collection and Display

Sampling methods can be complex, and the goal is to make a sample large enough and diverse enough to provide a basis for **confident predictions** based on it. This is how polls work. When a poll is declared to be "accurate 80% of the time nine times out of ten" the degree of confidence one can assign to the results of the poll is mathematically determined.

Of course, if it is realistic to sample every member in a population, then statistical methods are not necessary. Statistical methods are very important when large populations are involved.

Example: Response times to fires in hamlets

When we want to compare populations, for example fire department response times in Hamlets, or the time needed to complete a claim at WCB, we are interested in averages and the range of the data. Let's assume that the GNWT Department of Municipal and Community Affairs is reviewing fire departments in Hamlets. The time taken to respond to a fire is one indication of how effective a fire department is and will form part of the basis for deciding how to improve fire protection services.

Many other factors will also play a role, including resources, size of the community, presence of fire causing hazards, ability of firefighters, and communication/alarm systems in the community. Even before the results are in, we can be aware that a high response time does not by itself tell us why the response time is high or what is needed to shorten it.

Statistical analysis can provide data for decisions, but it cannot make decisions for us.

A statistical approach could include comparisons between response times in the NT and in similar jurisdictions. Observations, interviews, questionnaires, and surveys could all play a part in a strategy to gather data for analysis. At some point a decision could be made about a standard that all response times must satisfy. This standard could be the average response time that is observed across the territories or some other criterion.

Unit 3 – Statistics and Probability

Topic 1 – Data Collection and Display

Pros and Cons

Data can be collected in several ways. Here are some of the pros and cons

1. **Surveys/Questionnaires**

   Examples: telephone use customer survey, customer satisfaction survey for clients of a dealership.

   Pros: can be easily distributed by mail, email. Responses can be easily tabulated and analyzed with a computer program. Can be conducted by telephone and interview as well as by mail. Can ask specific questions that interest the researcher. Can be directed at a representative sample of a population.

   Cons: Not everyone who is given a survey will respond to it. A survey can be biased by literacy/technology. Only those who are able and motivated will complete the survey. The survey may fail to ask important questions related to its overall purpose. It may be difficult to distribute to a representative sample.

2. **Interviews**

   Example: Consumer research phone interviews. Community meetings.

   Pros: the researcher knows exactly who responds and can control the sample to make sure it is representative. The interview/meeting can thoroughly explore all aspects of a problem. For example, the long response time to a fire could be identified in an interview or meeting as to its cause, consequences, follow up and recommendations for the future. In small samples, such as are often found in Northern communities, interviews and meetings are very effective and practical.

   Cons: the interview can be subjective and leave it up to the interviewer as to what is recorded. Interviews can be time consuming and expensive. A small number of interviews can bias a sample. Not everyone who should be consulted will agree to an interview. Similarly in a meeting, the meeting can be dominated by one or two personalities and free and open discussion of an issue can fail to occur. The agenda may not be followed, and important data may not surface.

3. **Observations**

   Example: an air hose across Franklin Street counts the number of vehicles.

   Pros: these observations are objective and quantifiable.

   Cons: The sample must be controlled for weather, time of year, and other factors that can make the data fail to be representative.

   Once data is collected, it can be displayed using the graphs studied earlier in this curriculum. An additional type of graph is the circle graph, or "pie chart", where parts of a whole are broken into fractions. For example, a budget can be displayed on a circle graph showing that 20% of monthly income goes to clothing, 50% to rent and so on until 100% is accounted for.

Unit 3 – Statistics and Probability

Topic 1 – Practice Exam Questions

Question 1

Sally spends $45.00 a month on entertainment. She earns $1250 a month. What part of a circle graph would be needed to show the entertainment part of her monthly budget?

a) About 5%

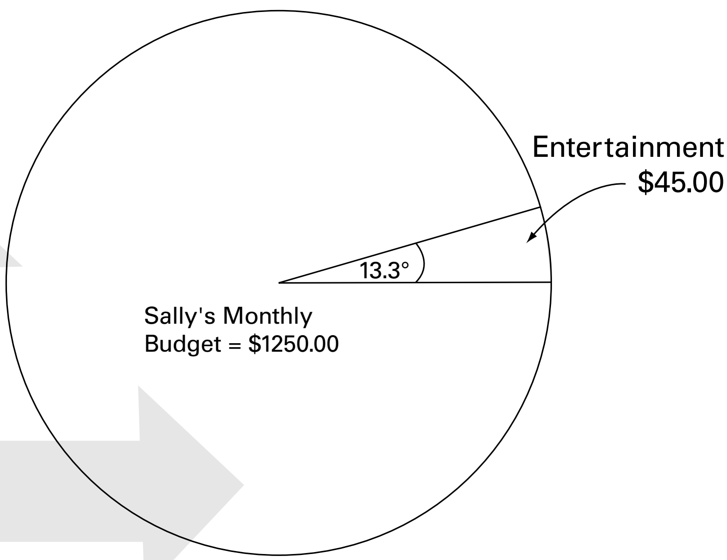
b) About 1/27

c) About 45%

d) About one third

Answer: b

Explanation
A circle graph assigns wedge shaped sectors to the parts of a whole. Here the whole = $1250, and the part is $45.00. The part to whole relationship is .036 or 3.6%. This can be rounded to 1/27 when you simplify 36/1000. The fraction form is useful for dividing a circle into a number of equal sized pieces that can be shaded in. If sally pays $450 in rent, that can be scaled to 27ths and shaded in proportionately. $450 dollars would be represented by almost 10/27. In general, the smallest amount being represented will determine the denominator for the circle graph's division. A computer program will do this automatically. If you are sketching the circle graph, find 1/27 x 360° = 13.3°. Then you can use a protractor to measure the angle and shade in the area subtended by the arc.



Topic 1 – Practice Exam Questions

Question 2

Which of the following is an example of a representative sampling method

a) Interviewing people who sign up for a survey

b) Phoning people who have last names ending in r

c) Measuring the height of every third person who enters a building

d) Stopping to interview people you are attracted to

Answer: c

Explanation
Only c will capture data that, over a large enough sample, is representative of a variable we are interested in, here the height of people entering a building.

Unit 3 – Statistics and Probability

Topic 2 – Central Tendency

Once data has been gathered for a population based on a question, we can proceed to analyze it in terms of its range, and the distribution of the data. We begin by defining the mode, median, and mean for a set of data. These are called measures of **central tendency**.

Suppose we want to develop a standard for the response time that northern fire departments should be able to satisfy. We could begin by gathering information on the response times of fire departments across the territories.

For example, we might learn that response times in minutes for the last 10 fires in a community were:

1. 12
2. 5
3. 32
4. 8
5. 17
6. 22
7. 10
8. 6
9. 27
10. 12

This data has a range from 5 to 32. Several measures of central tendency will help us to understand how to interpret this information. When this analysis is completed for a number of communities, we will have information that will allow us to compare departments and begin to discuss standards.

The Arithmetic Average is the Mean

This average is found by adding the values in a list of numbers and then dividing the sum by the number of items in the list. In statistical notation it looks like this:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\bullet^{x}}{n}$$

$\bar{x}$ is the symbol for the mean value of a list of values.

The mean for the table just given is 151÷10 = 15.1.

The Most Frequent Value is the Mode

The only value that occurs more than once is 12. In this list, 12 is the mode.

Unit 3 – Statistics and Probability

Topic 2 – Central Tendency

The Midpoint is the Median

The median is the midpoint value found by arranging the data from lowest to highest. If the number of items is even, the mean of the two values surrounding the midpoint is taken as the median. We reorder the list in terms of increasing response time to find the median:

1. 5
2. 6
3. 8
4. 10
5. 12
6. 12
7. 17
8. 22
9. 27
10. 32

This list has an even number of entries. We take the mean of items 5 and 6.

$$\frac{12 + 12}{2} = 12$$

The median response time is 12. The median is also a kind of average. In general usage, when the average is asked for the mean is meant. When the median is desired it should be stated explicitly.

Central Tendency

The way data clusters or does not cluster around a mean value, a mode, or a median gives important information. Each measure has its purposes. Consider the effect on each measure of central tendency if our list of response times was changed in certain ways.

1. Add extreme values

If a list of data has one or two values that are very far apart from the rest it can mean that the median is more representative (i.e. typical) of a set of data than the mean. Consider this list of response times where the highest value is no longer 32, but 120 minutes. (all other values remain unchanged). A response time of 120 minutes could happen if a fire broke out when the fire truck was not able to start.

1. 5
2. 6
3. 8
4. 10
5. 12
6. 12
7. 17
8. 22
9. 27
10. 120

Mean = 239/10 = 23.9

Mode = 12 (12 occurs more than once)

Median = 12 (12 is the mean of the middle entries: (12 + 12) ÷ 2 = 12

You can see that the mean response time is "skewed" or pulled away from the median by the extreme value. The median of 12 minutes is more typical of the time it takes to respond to a fire. In surveys, extreme values are sometimes discarded to avoid pulling the data away from the central tendency.

2. Add or subtract entries that repeat

It is sometimes useful to look at a table of values and simply pick the one that occurs most often as the "average". For example, if 8 apprentices in a class of 15 got 80% on a quiz, an instructor might be justified in saying the "average" score was 80%. The disadvantage is that we don't know the highest and lowest scores, and we don't know how the other 7 scores are distributed. They could all be lower than 80% or higher than 80%.

Sometimes there is more than one mode:

23, 14, 5, 23, 14, 67, 32,

In this list, there are two modes because both 23 and 14 occur twice.

Sometimes there is no mode

23, 14, 6, 17

No number occurs more than once. Since no number occurs more than once, this list has no mode.

3. Gaps in the data can change expected averages

A "gap", i.e. a range of values that one expects to find but that are missing, can also pull an average away from an expected value. For Example, in 1919 there was a gap in the number of men aged 19-24 in the population of England due to WWI. This gap had the effect of making both the median age and the mean age in that population much higher than would be expected. Notice the effect of a gap on average response times. Consider what happens if we replace values near the median with more extreme values to create an absence of any response times between 10 minutes and 22 minutes.

1. 5
2. 6
3. 8
4. 10
5. 22 (gap: no values between 10 minutes and 22 minutes)
6. 22
7. 45
8. 60
9. 100
10. 120

mean = 398/10 = 39.8

mode = 22

median = 22

Here the mean is higher than would be expected if there were no gap.

Box and Whisker Plots and Quartiles

A simple graphing technique will allow you to see the distribution pattern in a list of data. Box and whisker plots divide the data in a list into quartiles. Half of the numbers will be in the box, (quartiles 2 and 3 are in the box) and one fourth with higher values will be outside the box to the right, (the fourth quartile) and one fourth with lower values will be outside the box to the left. (the first quartile).

The boundary between the first quartile and the second is found by taking the mean of all the values below the median of the data. Likewise for the boundary between the third quartile and the fourth: take the mean of the values above the median of the data. These boundaries define the ends of the "box".

There will be four quartiles: the first with values lower than the middle number of those less than the median, the second with values in the box that approach the median from below, the third with values in the box greater than the median but less than the middle number of those greater than the median, and the fourth quartile with values greater than the middle number of those that are greater than the median.

Step 1: find the end points in the range and mark them with dots to mark the ends of the " whiskers" or values that will lie outside the box containing the central values.

Step 2: find the median and make a vertical line that will lie inside the box.

Step 3: find the middle number of the values greater than the median and the middle number of those less than the median. Mark this with vertical lines for the ends of the box and draw the box.

Example

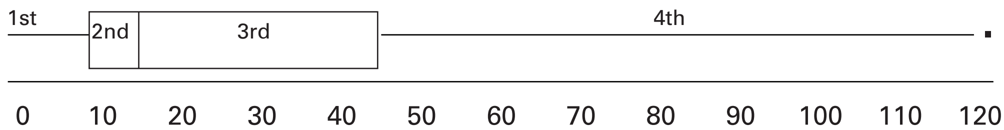
Make a box and whisker plot for the list of response times given above in (1):

1. 5
2. 6
3. 8
4. 10
5. 12
6. 12
7. 17
8. 22
9. 27
10. 120

Mean = 239/10 = 23.9

Mode = 12

Median = 12



From this box and whisker plot you can see the median = 12 and the range of response times that fall within the box are from 7.25 to 46.5. These values are the result of finding the mean value of the numbers greater and less than the median. This display does a better job of showing how the data is distributed than a mean would.

This method shows where most of the data cluster, i.e. within the box where half of the data is contained. This method allows us to see that the "whisker" on the right skews the central tendency away from where most of the data is. We can see that only one fourth of the data is located to the right of the box. We can also see that since one half of the data is inside the box, the mean between the boundaries of the box gives a good measure of central tendency, or average. Here that value is 7.25 + 46.5 divided by two, which is 26.87. This value is about 12.5% higher than the arithmetic mean (23.9) of the data.

Example:

1. Calculate the mode, median and mean for the following data

   12, 3, 14, 78, 5

   a) the mean = the sum divided by 5. 112/5 = 22.4

   b) the mode = the number that occurs most frequently. Here there is no mode.

   c) the median = the middle number in an order listing of these numbers: 3, 5, 12, 14, 78. 12 is the middle number in this odd number of items.

Unit 3 – Statistics and Probability

Topic 2 – Practice Exam Questions

Question 1

A firm has 6 secretaries earning $25,000, 2 managers earning $40,000, and an owner who earns $300,000. What measure of central tendency would not give a representative average for this data?

a) The mode

b) The median

c) The mean

d) The range

Answer: b

Explanation
Make a list of the data from lowest to highest values and calculate the three averages:

1. 6 x $25,000
2. 2 x $40,000
3. 1 x $300,000

Mean = $530,000/9 = $58,888

Median = $25,000

Mode = $25,000

In this list, the owner's salary "skews" the data. Most employees earn $25,000 and this is also the median salary. The most frequent salary is also $25,000. The mean, however is greater than all salaries except the owner's. It does not give a good measure of central tendency.

Question 2

The mode of a set of data is 14. There are 22 values in the data. What can you conclude?

a) The number 14 is in the list at least twice

b) 14 is halfway down the list

c) 14 is the midpoint in the range

d) 14 is the average of the data

Answer: a

Explanation
If a list has a mode, it is a number that must appear at least twice. The other choices could be true, but we don't have enough information to know.

Topic 2 – Practice Exam Questions

Question 3

What values will define the ends of the box in a box and whisker plot based on the following data: 4, 16, 72, 39, 5

a) 20 and 39

b) 4.5 and 55.5

c) 27.2 and 41.5

d) 18 and 55.5

Answer: b

Explanation
The ends of the box correspond to the boundaries between the 1st and second quartiles, and the third and fourth quartiles. The median is 16, which is the third value in a list of five values when the list is arranged in ascending order. The mean of the values above the median is 55.5, and the mean of the values below the median is 4.5. These are the values that define the ends of the box and separate the quartiles.

Unit 3 – Statistics and Probability

Topic 3 – Probability

Background

Probability is closely linked to statistical analysis. Consider the following conclusions that can be drawn about the probability of an individual in a population landing in a predictable place in a statistical description of that population:

1. If you were born after 1960 you are more likely to be a single parent than if you were born before 1960.
2. The likelihood that you will live more than 70 years has increased by 30% over the last 20 years.
3. If you are a woman, you stand a good chance of being paid less than a man for the same work.

Population characteristics allow us to make predictions like these. Although there is no certainty that an individual will fulfill these predictions even though they belong in a specified population, they are based on generalizations that are supported by the characteristics of populations that have been studied.

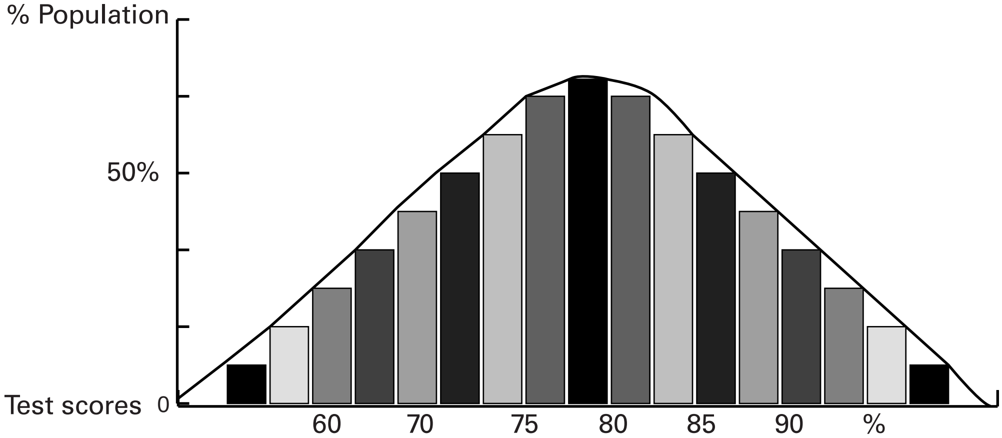
Society depends on statistical analysis to look at the probabilities of future outcomes. For example, if the number of days that workers are calling in sick is increasing by 4% per year over the past four years, an alarm should go off to alert agencies and individuals to what is becoming a public health and economic issue. Two key points can guide you on the exam:

1. **Averages can be used as norms to detect deviance or an emerging trend.**

   Example: Taking 12 sick days per year is the average. Anyone taking more than this number may have health issues that need special attention. In this sense they are "abnormal" or exceptions to an expectation based on data.

   A bell curve, or normal curve is used to represent many kinds of populations where the data clusters around a mean value that is representative of the central tendency. These graphs are the smooth result of histograms (bar graphs) that show the relative frequency of a variable being measured across a population.

   In the next diagram, an ideal picture is given of a normally distributed range of test scores. The larger the population, say all grade 12 students in Canada, the smoother the curve should be. The mean score is also the "normal" score and creates a standard for future expectations of the same population. This is the idea behind standardized testing. If a nationally standardized test is taken in the Northwest Territories then people taking the test in the NT should show a normal curve distribution of scores. If the results are skewed right we can conclude that people in the NT are doing better than the average, and the reverse if the results are skewed left.



Here most scores cluster between 75% and 85%. You can see how a curve of this kind could show the frequency of relative heights over a population, or of income levels in a certain kind of society. It could also describe the height of trees in a forest, snowfall amounts in a winter, or response times in fire departments across the country.

Because this fits our expectations of a normal distribution, we can use it to think about abnormal distributions. For example, if a test is given and most scores are between 50% and 60%, we might conclude that something is "wrong". Perhaps the test was too hard, the material wasn't taught, or something interfered with performance on the test.

2. **Norms for a population do not predict the behavior of individuals, only the likely behavior of groups of individuals.**

There are many problems associated with polls and polling. Some were mentioned earlier in the section on the pros and cons of data collection. Keep these in mind as you work with statistical information: consider the source, consider how representative samples are, and consider whether the right questions are being asked. For example, you could read that a Northern community has 70% unemployment and this is "abnormal" because the rest of Canada has a rate of only 12%. However, on closer inspection it could turn out that there are no jobs in this community, and the fact that 30% of the workforce is employed is a minor miracle. Similarly, you might read that the illiteracy rate in the NT is 40%, but on closer inspection it turns out that those who cannot read never had an opportunity to learn and/or are in a specific age group, for example people over 70 years old. It is essential to know what population is being sampled, as well as to understand the possible causes and context for a conclusion based on a single comparative statistic.

Topic 3 – Probability

Finding Probability Numbers: Independent Events

What is probable is what is most likely to happen. A **probability number** is an indicator between 0 and 1 of how likely an event is- but a particular event can never be predicted exactly using a probability number unless it equals one. If an event is impossible, it has 0 probability. If an event is certain, it has 100% probability or a probability of 1. For example, the probability that no number is equal to itself is 0, and the probability that a number exists that is odd is 1. These are certainties.

Notice that probabilities can be very small (very unlikely) or very large (very likely). For example, the probability that an asteroid will hit the earth tomorrow is very small – but not impossible. The probability that you will blink your eyes in the next hour is very high – but not 1(i.e. not certain).

Percentages Can Represent Probabilities

If 50% of the apprentice class is female then the probability that a random selection of one person from the class will be female is .5.

If 20% of apprentices in the class smoke, then the probability that a random selection from this group will produce a smoker is .2.

The probability of selecting an apprentice who is female and who smokes is .1 (10% = 20% of 50%)

One way to find the probability of an event, known as the **Monte Carlo method,** is to conduct a large number of experiments and record the results. The results of a large number of experiments will give the probability of an event. For example, a coin toss experiment was conducted 20,000 times by a mathematician in 1900. He recorded 10,012 heads and 9,988 tails. We can conclude that the number of heads will approach equality with the number of tails as the number of trials increases. However, it is possible to observe 3 or even four heads or tails in a row using a fair coin. **Uncertainty is part of probability theory and probability numbers cannot predict the future except in a general or longer term fashion.** The Monte Carlo method is an experimental approach to probability. A theoretical approach looks at ideal situations and the limit that defines a probability.

The gap between results that are predicted for a large number of experiments and those that are actually obtained experimentally can be very informative. For example, one might predict that rats will divide equally when they come to a fork in a maze, and that the probability of a rat taking the right fork is 1/2. An experiment might reveal that 1/3 is the probability. This would lead to new experiments that investigate why rats prefer left turns more than right turns. It is unclear that outcomes are equally likely until experiments confirm this hypothesis in many situations.

Topic 3 – Probability

Events for which the outcome cannot be predicted, i.e. they are uncertain, are called **experiments**. We only know in hindsight how an experiment turned out. For example, a fair coin or a fair die, will produce each of the outcomes that are possible for a coin and for a die with the same probability. Each toss, or experiment, has no history and will not be influenced by the previous toss. A head is just as likely as a tail on a particular toss. These events are independent. In these examples we say that the outcomes in the sample space of a coin or a die are equally likely. The same is true of a fair lottery: every ticket has an equal chance of winning. The probability that a particular ticket will win equals the number of winning tickets divided by the total number of tickets issued.

P = 1/n

In the case of a die, there are 6 equally likely outcomes, and so we say the probability of any one of them is 1/6. We can generalize and say **that if n events are equally likely, then the probability of any one of them occurring is 1/n.** For example in a well shuffled deck any one of the 52 cards is equally likely to be the first card dealt. The probability of getting, say , the ace of spades, will be 1/52.

The **sample space** for a die is 6 and for a coin 2. The sample space contains all of the possible outcomes of an experiment. **An event is a subset of a sample space.** When we think of the probability of getting a head on a coin toss, we are thinking about an event in the sample space of the coin tossing experiment. The letter E is used for a favourable outcome, i.e. the outcome in the sample space that we are interested in.

Set theory notation is useful in discussing probability. A coin has two possible outcomes, heads or tails, both are equally likely, and so we say the chances (or probability) of either a head or a tail is 1/2 for each **trial** (or use of the coin).

This result can be generalized in a definition:

$$P(E) = \frac{n(E)}{n(S)}$$

The probability of event E, written P(E), when E is a subset of the sample space S, is the number of equally likely outcomes for the event we are considering, n(E), divided by the number of equally likely outcomes in the sample space to which the event belongs, n(S).

An event must be part of the sample space. When the event has only one outcome it is a simple event. Tossing a head is a simple event. Throwing a number greater than 2 on a die is a complex event because 3, 4, 5 or 6 will satisfy it.

P(E) = 1/2 for the event of getting heads or tails with a coin. P(E) = 1/6 for the event of getting a particular number from 1-6 on the toss of a die. In the case of a coin, S = {H, T}, meaning that the sample space of a coin has two equally likely possible outcomes, either a head or a tail. In the case of a die, S = {1, 2, 3, 4, 5, 6}.[26]

[26] You may want to review the discussion on fractions in module one: foundations. The interpretation of a numerator as the portion of a divided whole we are thinking of, applies to the meaning of fractions in probability.

Topic 3 – Probability

The event of getting a head on a coin toss can be expressed as E = {H}, and the event of getting a 6 on the throw of a die can be expressed as E = {6}.

Examples

1. What is the probability of getting a number smaller than three when rolling a die?

   First identify the sample space for a die and then the number of items in the sample space:

   S = {1, 2, 3, 4, 5, 6} and n(S) = (6)

   Next identify the elements in the sample space that satisfy the event outcomes we are interested in . Here only 1, and 2, are less than three in the sample space for a die. Therefore: E = {1, 2} and n(E) = 2

   Use the definition of probability given above to find P(E):

   $$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

2. What is the probability of getting at least one tail if you toss a pair of coins?

   First identify the sample space (all possible outcomes) for two coins:

   S = {(H, T), (T, H), (H, H), (T, T)} this tells us that n(S) = 4

   Notice that (H, T) and (T, H) are the results of different tosses, they are not the same outcome counted twice.

   Next define the set of outcomes that satisfy the event we are interested in, here getting one tail. Notice again that E is contained in (i.e. is a subset) S.

   E = {(T, T), (T, H), (H, T)} and n(E) = 3, therefore:

   $$P(E) = \frac{n(E)}{n(S)} = \frac{3}{4}$$

The Probability of Two Independent Events

More complex events combine different subsets of a sample space. In set theory, the operations of **union, intersection, and complement** can help define the probability of more than one independent event happening in the same experiment.

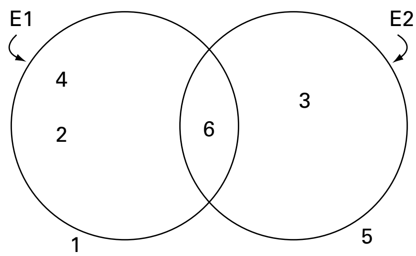
Consider how a table can show all possible outcomes of two independent events. Let the first event (E1) be that the die shows an even number, and the second event (E2) be that the die shows a 3 or a 6. A table can show which outcomes in the sample space of a die satisfy one or both events:

	1	2	3	4	5	6
E1		℘		℘		℘
E2			℘			℘

You can see that only the outcome "6" will satisfy both events. 6 is the only outcome common to both events. 6 will satisfy either event. 6 is the only element in the **intersection** of E1 and E2. No other elements in the sample space appear in both E1 and E2. The probability of E1 = 3/6 = 1/2, and the probability of E2 = 2/6 = 1/3.

We can connect the probabilities of E1 and E2 by looking at their union and their intersection. If we are interested in which outcomes satisfy one or both events, we are interested in their union. **The union of two sets includes all numbers that are in either set or in both sets.** Now we look at the table and see that there are 4 outcomes that would work to satisfy this complex event. These four outcomes, 2, 3, 4, 6, are the union of E1 and E2. A union of two sets has all the elements that appear in either or both sets. These are the elements common to both. E1 ˘ E2 (the union of E1 and E2) = {2, 3, 4, 6}. Notice that we did not count 6 twice because throwing it once already satisfies both E1 and E2 and it is equally likely on any toss. Notice also that the union of these events is also a subset of the sample space for a die.

This Venn diagram shows where the elements in the sample space belong in our



example. Notice that 1, and 5, are outcomes that satisfy neither event. 6 is the only number that satisfies both events. 6 appears in the region shared by the two sets, E1 and E2. 4 satisfies E1 but not E2, and 3 satisfies E2 but not E1.

The set of {2, 3, 4, 6} is the set that satisfies one or both events. This set has four elements, so n(E1 » E2) = 4. A second, but different complex event is the intersection of E1 and E2. The set of {6} satisfies both events and is the only element in the intersection. This set has one element so n(E1 « E2) = 1. Use diagrams of this kind to help you understand the outcomes for combined events.

We can use our definition to see that the probability of throwing a number on a die that satisfies E1 » E2 is .66. P(E1 » E2) = n(E1 » E2) / n(S) = $\frac{4}{6}$ = .66.

This means that there is a 2/3 probability that any toss will produce either a 2, 3, 4, or 6.

Use the same definition to see that the probability of throwing a number that satisfies E1 « E2 = 1/6 = .166.

P(E1 « E2) = n(E1 « E2) / n(S) = $\frac{1}{6}$

This leads us to a rule for the probability of complex events that are the union of simple events.

The Union Rule for Probability

If A is "dealt a face card" and B is "dealt a club", then A » B is "dealt a face card or a club or a face card that is a club". All face cards and all clubs will satisfy.

The fact that some face cards are also clubs, means that we have to subtract the probability of getting a face card that is also a club to avoid counting this outcome twice. This leads to the union rule for probability that we already incorporated in the above discussion on not counting six twice in the union of E1 and E2.

P(A » B) = P(A) + P(B) – P(A « B)

Example:

If a single card is drawn from a deck of 52 cards, find the probability that it will be either black, or a face card, or both. (Consider an ordinary deck with jack, queen, king as the face cards in each of four suits.) There are 26 black cards, and 26 red cards.

We need to describe two events: B = black, F = face card. You can see that the probability of a black card = P(B) = 26/52. There are 12 face cards so P(F) = 12/52. There are 6 black face cards in the deck and this is the intersection of B and F. P(B « F) = 6/52 . Now find the probability of B » F using the rule that subtracts the intersection so we don't count it twice:

P(F » B) = P(F) + P(B) – P(F « B)

$$= \frac{12}{52} + \frac{26}{52} - \frac{6}{52} = \frac{32}{52} = .615$$

You can use the union rule for probability to reach the same conclusions for the union of E1 and E2:

P(E1 » E2) = P(E1) + P(E2) – P(E1 « E2)

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = .66$$

Unit 3 – Statistics and Probability

Topic 3 – Probability

Here is a summary of the operations on any two independent events (A and B) in a sample space S:

1. A « B is satisfied when **A and B** are satisfied.

   If A is "dealt a face card" and B is "dealt a club", then A « B is "dealt a face card that is a club" Only the jack, queen, and king of clubs will satisfy this.

2. A » B is satisfied when **either A or B or both** are satisfied.

   If A is "dealt a face card" and B is "dealt a club", then A » B is "dealt a face card or a club or a face card that is a club" all face cards and all clubs will satisfy.

3. A′ is satisfied, i.e. will occur, when A does not. A′ is the **complement** of A. Example: if a head is not the result of tossing a coin, then the complement, a tail will be the result. **The probability of an event and its complement will**

Example

What is the probability that the sum of numbers rolled on two dice will not be 7?

The sample space for this experiment has 36 pairs of numbers. Of these, the ones that equal seven are fewer and easier to find. Do this, and then solve an equation setting the sum of this event and its complement equal to one. The probability of the complement is what we want.

There are six combinations that add up to seven: (6 +1), (5+2), (4+3), (3+4), (2+5), (1+6). The probability of getting a sum = 7 is 6/36 = 1/6.

1/6 + (probability of getting a sum not equal to 7) = 1 (because one of the two outcomes must occur).

Probability of not getting a sum equal to 7 = 5/6.

Unit 4

Practice Exam Questions for Math – Module 5 – Special Topics

Each topic in the table of contents has sample questions to test your preparation for the trades entrance level five exam. You should aim for 100%, and study the sections of the curriculum for any topics that you do not get right. Turn to the appropriate section of the curriculum whenever you need help.

The trades entrance special topics math curriculum is based on "need to know" competencies that are important in trades requiring the level five exam. You may want to use the following sample exam questions both as a way of assessing what you need to learn before you work on the curriculum, and as a test of what you know after you have completed your preparation for the exam.

Answer Key

After the questions for each topic you will find an answer key. The following questions are grouped in clusters of related items. You may want to randomly pick questions from different parts of the test for pre-test purposes, or you may want to home in on a particular area. If you want to take a pre-test, select every 2nd or fourth question for example, then score yourself, and you will get an idea of what to study. After studying, try every first or third question and see how well you do and repeat the study cycle based on your results.

Unit 4 – Practice Exam Questions

Topic 1 – Functions

Question 1

Which ordered pair is a solution for $y = 5x + 2$?

a) (100, -2)

b) (20, 102)

c) (-20, 10)

d) (-20, -100)

Answer: b

Question 2

A scatter plot shows that the time it takes for an ice jam to break up decreases when the weather is clear and the sun is shining. Which choice describes the functional relationship that is implied by this data ?

a) Break up time as a function of temperature.

b) Temperature as a function of break up time.

c) Rate of break up as a function of sunshine.

d) Rate of temperature change as a function of time needed for break up.

Answer: c

Question 3

A table for $y = 3x^2 – 5$ uses the output from each row as the input for the next row. If the input for row 3 is 10, what will be the input for row 4?

a) 250

b) $5 + \sqrt{5}$

c) 295

d) 25

Answer: c

Unit 4 – Practice Exam Questions

Topic 1 – Functions

Question 4

What is the correct value needed to complete this table?

x	y
10	25
5	15
20	45
100	205
3	?

a) 11

b) 15

c) 12

d) 18

Answer: a

Question 5

What is the equation for the function given by the table in the last question?

a) y = x/2

b) y = 2x + 5

c) y = 2x + 2

d) y = 3x/2

Answer: b

Question 6

Which of the following has one output value for several different input values?

a) A person's salary as a function of hours worked.

b) Interest income as a functin of time that principal is invested at a fixed rate.

c) The height of a person as a function of their age.

d) The amount of income tax as a function of income.

Answer: d

Unit 4 – Practice Exam Questions

Topic 2 – Trigonometric Functions

Question 1

What is the exact value of x on the interval 0 ″ x ″ 2≠ in the equation
4 sin x – $\sqrt{3}$ = 2 sin x?

a) ≠/2, 2≠/3

b) ≠/3, 2≠/3

c) $\frac{\sqrt{2}}{2}$

d) ≠, 2≠

Answer: b

Question 2

Find q if 0 ″ q ″ 360° in 3 sin q – 5 = -2 sin q

a) 45°

b) 30°

c) 60°

d) 90°

Answer: d

Question 3

Which identity is true

a) $\cos^2 x - \sin^2 x = 1$

b) $\sin^2 x - \cos^2 x = 1$

c) $\sin^2 + \cos^2 = 1$

d) $1 - \sin^2 x = \frac{\text{Opposite}}{\text{Hypotenuse}}$

Answer: c

Unit 4 – Practice Exam Questions

Topic 2 – Trigonometric Functions

Question 4

What is the period of the sine function?

a) Can’t tell without knowing the amplitude

b) $\neq$

c) $\neq/2$

d) $2\neq$

Answer: d

Question 5

Which angle is co-terminal with 45° ?

a) 365°

b) 145°

c) 405°

d) -45°

Answer: c

Question 6

An electric current described by a sine wave reaches a minimum value of -6 and a maximum value of 4. What is the amplitude?

a) 5

b) -5

c) 10

d) 3

Answer: a

Unit 4 – Practice Exam Questions

Topic 2 – Trigonometric Functions

Question 7

Which equation describes a cosine with a period of ≠/2?

a) y = cos 4x

b) y = ≠/2 cos x

c) y = cos x + ≠/2

d) y = 4 cos x

Answer: a

Question 8

What is the solution for sin2q – cosq = 0 for 0 ″ q ″ 360°?

a) 45°, 90°, 135°, 180°

b) 30°, 90°, 150°, 270°

c) 60°, 120°, 180°, 240°

d) 0°, 360°

Answer: b

Question 9

What is the general solution for sin x = -1 ?

a) x = 0

b) x = ≠ and ≠ + k2≠ when k is any positive integer

c) $\{x \mid x = \frac{3\neq}{2} + 2k\neq\}$

d) $\{x \mid x = \frac{\neq}{2} + k\neq\}$

Answer: c

Unit 4 – Practice Exam Questions

Topic 2 – Trigonometric Functions

Question 10

What is the radian measure of 135°?

a) 2.36

b) ≠

c) 14.4

d) ≠/4

Answer: a

Question 11

Which line is not an asymptote for the tangent function?

a) x = -≠/2

b) x = ≠/2

c) x = ≠/4

d) x = 3≠/2

Answer: c

Question 12

Which equation describes a sine wave that has a phase shift ≠/2 to the left?

a) y = ≠/2 sin x – 1

b) y = sin x – ≠/2

c) y = sin x + ≠/2

d) y = 2x + ≠

Answer: c

Unit 4 – Practice Exam Questions

UNIT 4

Topic 2 – Trigonometric Functions

Question 13

Which identity can be used to find the exact value of cos 75° without a calculator?

a) sin 2x + cos 2x = 1

b) sin 2x = 2 sinx cosx

c) cos (a + b) = (cos a)(cosb) – (sin a)(sin b)

d) sin 2x = 1 – 1 cos 2x

Answer: c

Question 14

What are the solutions for 0° ″ q ″ 360° in 2 $\cos^2$q + sinq – 1 = 0?

a) 30°, 60°, 90°, 270°

b) 60°, 120°, 180°, 240°, 300°, 360°

c) 0°, 180°, 360°

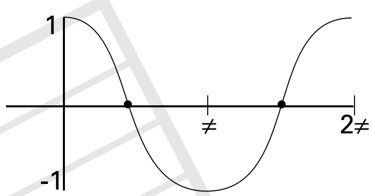
d) 45°, 120°, 360°

Answer: c

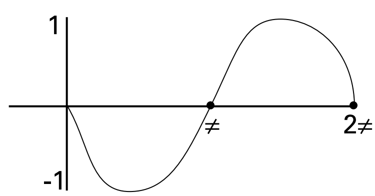
Question 15

Which graph shows one period of y = cos x?

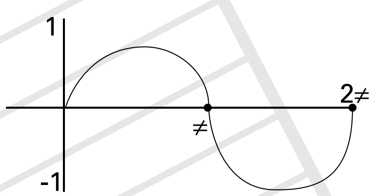
a)



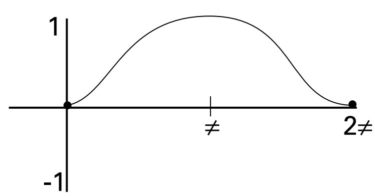
c)



b)



d)



Answer: a

Unit 4 – Practice Exam Questions

Topic 2 – Trigonometric Functions

Question 16

Which angle equals 12 radians?

a) 120°

b) 360°

c) 687.5°

d) 2160°

Answer: c

Question 17

What is the reference angle and sign of the cosine for 210°?

a) 30°, negative

b) 30°, positive

c) 60°, negative

d) 60°, positive

Answer: a

Question 18

What is the exact value of csc 300°?

a) –1/2

b) $\frac{\sqrt{2}}{2}$

c) $\frac{\sqrt{3}}{2}$

d) $-\frac{2}{\sqrt{3}}$

Answer: d

Topic 2 – Trigonometric Functions

Question 19

Find cos 2B if B is in quadrant two and sin B = 3/5

a) 1/2

b) 7/25

c) 1/3

d) 2/15

Answer: b

Question 20

If y = sin x has a phase shift of -≠/2, what will be the x coordinate after one period beginning at x = -≠/2 ?

a) -2≠

b) 3≠/2

c) ≠

d) 5≠/6

Answer: b

Question 21

A circle has a radius of 20 feet. How long will the arc be that is intercepted by a central angle of 30°?

a) 10.47 feet

b) 60 feet

c) 3.33 feet

d) 29. 2 feet

Answer: a

Unit 4 – Practice Exam Questions

Topic 2 – Trigonometric Functions

Question 22

A point on the circumference of a circle has the coordinates (.96, -.26). What quadrant is the point in?

a) one

b) two

c) three

d) four

Answer:d

Question 23

Which equation has no solution?

a) sin x = ≠

b) cos x = 1/2

c) tan x = 0

d) sec x = 1

Answer: a

Question 24

What are the coordinates of the point on the unit circle that the radius touches after rotating ≠/4 radians?

a) (.5, .5)

b) $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$

c) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

d) (1/4, 1/4)

Answer: c

Unit 4 – Practice Exam Questions

Topic 2 – Trigonometric Functions

Question 25

The graph of a periodic function has (2, 8) and (5,8) at the ends of one cycle. What will be the y coordinate when x = 11?

a) 7

b) 8

c) 6

d) 11

Answer: b

Unit 4 – Practice Exam Questions

Topic 3 – Exponential Functions

Question 1

What is x when f(x) = 60 in the equation $y = 4^{x+2} - 4$?

a) x = 3

b) x = 1

c) x = 0

d) x = 1/2

Answer: b

Question 2

Which ordered pair is on the graph of $y = (1/2)^x$?

a) (2, 8)

b) (-2, 4)

c) (1/4, 1/16)

d) (1, 1/2)

Answer: d

Question 3

Which choice correctly describes the graph of $y = 4^{x+3} - 5$?

a) An increasing function that approaches y = -5 as x decreases.

b) A decreasing function that approaches y = 5 as x increases.

c) An exponential equation that goes through (0,1).

d) An increasing function that is shifted 3 units right.

Answer: a

Unit 4 – Practice Exam Questions

Topic 3 – Exponential Functions

Question 4

A = P x e^{rt} is the formula for the amount of a sum that is compounded continuously at a fixed rate. A sum of $25,000 is invested for 5 years at 6%. What will the balance be after 4.3 years?

a) $38,650

b) $34,432

c) $33,746

d) $35,780

Answer: c

Question 5

$f(x) = \left(\frac{1}{4}\right)^x$. What is true about this function?

a) Increasing, domain (–° , °), range (0, °).

b) Decreasing, domain (–° , °), range (0, °).

c) Decreasing, domain (–° , 0), range (0, °).

d) Increasing, domain (–5, °), range (0, °).

Answer: b

Question 6

What is the equation for the graph of $y = 2^x$ after it has been moved 3 units to the left and then 2 units up?

a) $y = 2 + 2^{x+3}$

b) $y = 2^{x-3} - 2$

c) $y = 2^{x+2} - 3$

d) $y = 2^{3x} - 2$

Answer: a

Unit 4 – Practice Exam Questions

Topic 3 – Exponential Functions

Question 7

Solve this equation: $-3^x = -27$

a) -9

b) There is no real number solution.

c) -3

d) 3

Answer: d

Question 8

If $e^x = e^3$, what is x?

a) e

b) 1/e

c) 3

d) 2.713

Answer: c

Question 9

Solve $8^x = 2$

a) 8

b) 2

c) 1/2

d) 1/3

Answer: d

Unit 4 – Practice Exam Questions

Topic 3 – Exponential Functions

Question 10

Which ordered pair satisfies the equation $y = 2^x$?

a) (2,1)

b) (4,2)

c) (1,2)

d) (1/2, 1/4)

Answer: c

Unit 4 – Practice Exam Questions

Topic 4 – Logarithmic Functions

Question 1

Solve for x in $\log_2 x = 2$

a) –2

b) 0

c) 2

d) 4

Answer: d

Question 2

Find x in $\log_x 4 = 3$

a) $x = x^3$

b) $x = 4^{1/3}$

c) $x = \sqrt{2}$

d) $x = e$

Answer: b

Question 3

Find x in $\log_8 4 = x$

a) $x = 2/3$

b) $x = 1/3$

c) $x = \sqrt{2}$

d) $x = 1/4$

Answer: a

Unit 4 – Practice Exam Questions

Topic 4 – Logarithmic Functions

Question 4

Find x in $\log_3 x = -2$

a) $x = 3$

b) $x = 1/2$

c) $x = 1/3$

d) $x = 1/9$

Answer: d

Question 5

Which ordered pair is on the graph of $y = 3^{-x}$?

a) (1, 1/9)

b) (-1, 3)

c) (3, 27)

d) (3, -9)

Answer: b

Question 6

What is a logarithmic form for $3^x = 5$?

a) $\log_5 3 = x$

b) $x \log_3 5 = 5$

c) $5 \log_x 3 = 3$

d) $\log_3 5 = x$

Answer: d

Unit 4 – Practice Exam Questions

Topic 4 – Logarithmic Functions

Question 7

What changes are made to $y = \ln x$ in $y = \ln (x - 3) - 4$?

a) The graph is shifted 4 units left and 3 units down.

b) The graph is shifted 3 units right and four units down.

c) The graph is shifted 4 units up and 3 units right.

d) The graph is shifted 3 units down and 4 units left.

Answer: b

Question 8

Which expression is equivalent to $\log_{12} x$?

a) $\frac{\ln x}{\ln 12}$

b) $x \log_{12}$

c) $\log 12 - \log x$

d) $\ln x - \ln 12$

Answer: a

Question 9

The concentration of H+ ions in a solution (a measure of acidity) has a ph of 8. Ph is defined as -log (H+). What is the concentration of H+ (ignoring units)?

a) 10^{8}

b) 10^{-8}

c) .0001

d) 1/10

Answer: b

Unit 4 – Practice Exam Questions

Topic 4 – Logarithmic Functions

Question 10

Solve $A = P \times e^{rt}$ for t

a) $t = \frac{A}{P} \ln e^{r}$

b) $t = \frac{\ln p - \ln A}{r}$

c) $t = \frac{\ln A - \ln P}{r}$

d) $t = \frac{r\ln P}{A}$

Answer: c

Question 11

Solve this equation: $\log_3 x = 1/2$.

a) 3

b) $\sqrt{3}$

c) 9

d) 1/3

Answer: b

Question 12

When the graph of $y = \ln x$ is reflected across $y = x$, what will be the result?

a) $y = 2^x$

b) $y = \ln 2x$

c) $y = x^2$

d) $y = e^x$

Answer: d

Unit 4 – Practice Exam Questions

Topic 4 – Logarithmic Functions

Question 13

Which equation gives the exponential form of log (6) = x ?

a) log 6 = $(10)^1$

b) $2^2 + 2x = x$

c) log 2 + log 3 = x^2

d) $10^x = 6$

Answer: d

Question 14

What is the graph of the reflection across y = x of y = $\log_2 x$?

a) y = lnx

b) y = 2^x

c) y = 2log x

d) y = x^2

Answer: b

Question 15

Which answer describes the graph of y = $\log_{1/4} x$?

a) A decreasing function that approaches -° as x increases.

b) An increasing function with y = x as an asymptote.

c) An increasing function that approaches ° as x increases.

d) A decreasing function that approaches 0 from both sides.

Answer: a

Unit 4 – Practice Exam Questions

Topic 4 – Logarithmic Functions

Question 16

What does the graph of $y = \log_2 (x - 1)$ do to the graph of $y = \log_2 (x)$?

a) Moves it one unit up.

b) Moves it one unit down.

c) Moves it one unit left.

d) Moves it one unit right.

Answer: d

Unit 4 – Practice Exam Questions

Unit 1 – Topic 5 – Solve a Quadratic Equation

Question 1

What are the solutions for x in $x^2 + 6x + 2 = 0$?

a) $\frac{-6 \pm \sqrt{28}}{2}$

b) $-3 \pm 2\sqrt{2}$

c) $4 \pm 2\sqrt{7}$

d) There are no solutions.

Answer: a

Question 2

What are the solutions for x in this equation: $(3x - 2)(x + 2) = 6$

a) $\frac{-4 \pm \sqrt{136}}{6}$

b) $\frac{-3 \pm \sqrt{120}}{8}$

c) 2/3 and -2

d) $\frac{-4 \pm \sqrt{-136}}{-20}$

Answer: c

Unit 4 – Practice Exam Questions

Unit Two – Topic 1 – Vectors and Scalars

Question 1

Which quantity is a vector?

a) The speed of a river.

b) The direction of a caribou.

c) The salary of a worker.

d) The force and direction of a wind.

Answer: d

Question 2

A force of 3 lbs acts downward and a force of 5 lbs acts upward on an object. Which vector (**V**) gives the result?

a) **V** = 8lbs down

b) **V** = 6lbs up

c) **V** = 2lbs down

d) **V** = 2lbs up

Answer: d

Question 3

A vector describes the direction and force of wind at the Inuvik airport as 10mph east. What is the opposite to this vector?

a) 20 mph east

b) 10 mph west

c) 10 mph south

d) 10 mph east

Answer: b

Unit Two – Topic 1 – Vectors and Scalars

Question 4

A vector is multiplied by 5. What will happen to its direction?

a) It will reverse.

b) It will rotate.

c) It will stay the same.

d) It will increase.

Answer: c

Question 5

In a diagram, vector arrow A is three times longer than vector arrow B and points in the same direction. What single arrow can replace these arrows?

a) An arrow that is four times the length of B and points in the same direction as B.

b) An arrow that is three times the length of B and points in the same direction as B.

c) An arrow that is three times A + B and points opposite to A.

d) An arrow that is four times the length of B and points in any direction.

Answer: a

Question 6

What is the sum of $\mathbf{A} + 2\mathbf{A} - \mathbf{A}/2$?

a) $5\mathbf{A}/2$

b) $3\mathbf{A} - 2$

c) $2\mathbf{A} + \mathbf{A}/2$

d) $\mathbf{A}/2 - 2\mathbf{A}$

Answer: a

Unit 4 – Practice Exam Questions

Unit 2 – Topic 2 – Vector Operations

Question 1

A vector has a magnitude of 5000 and a direction angle of 45°. What is the horizontal component?

a) $2500\sqrt{2}$

b) $\sqrt{5000}$

c) $\cos 45° + \sqrt{5000}$

d) $y^2 - \sqrt{5000}$

Answer: a

For Questions 2 – 4

Let $\vec{V} = \langle 4, -2\rangle$, $\vec{S} = \langle -3, 6\rangle$, $\vec{R} = \langle 5, -2\rangle$

Question 2

find **S** + 3**R**

a) $\langle 18, 6\rangle$

b) $\langle 12, 0\rangle$

c) $\langle 15, -3\rangle$

d) $\langle 8, -10\rangle$

Answer: b

Question 3

Find **R – S**.

a) $\langle -1, -4\rangle$

b) $\langle -8, 8\rangle$

c) $\langle 8, -8\rangle$

d) $\langle 8, 8\rangle$

Answer: c

Unit 4 – Practice Exam Questions

Unit 2 – Topic 2 – Vector Operations

Question 4

Find $\vec{V} \cdot \vec{R}$

a) 16

b) 8

c) 28

d) 24

Answer: d

Question 5

A bush pilot flies 20 miles northwest and then 10 miles southeast. What is his position from where he started?

a) 15 miles northeast

b) 10 miles southwest

c) 10 miles northwest

d) 30 miles west

Answer: c

Question 6

What is the component form of **V** when $|\vec{V}| = 8$, q = 45°?

a) $\langle 4\sqrt{2}, 4\sqrt{2} \rangle$

b) $\left\langle .25, \frac{\sqrt{3}}{2} \right\rangle$

c) $\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

d) $\langle \sqrt{2}, \sqrt{2} \rangle$

Answer: a

Unit 4 – Practice Exam Questions

Unit 2 – Topic 2 – Vector Operations

Question 7

What is the linear combination of the unit vectors **i** and **j** that is equal to $\langle -5, -1 \rangle$?

a) 2**i** – 5**j**

b) -**i** + 5**j**

c) -5**i** – **j**

d) 5**j** + **i**

Answer: c

Question 8

If **V** = $\langle 3, 1 \rangle$ and **U** = $\langle -2, 3 \rangle$ what is the magnitude and direction angle for **U** – **V**?

a) 6.6, 33°

b) $3\sqrt{2}$, 47°

c) $\sqrt{69}$, 122.7°

d) $\sqrt{29}$, 158.2°

Answer: d

Question 9

A hunter paddles west at 2 mph across a river that is flowing 6 mph south. In degrees measured from due north clockwise, what will be the direction the hunter actually travels in?

a) 288°

b) 198°

c) 180°

d) 213°

Answer: b

Unit 4 – Practice Exam Questions

Unit 2 – Topic 2 – Vector Operations

Question 10

What is the magnitude of the resultant when two vectors of 3 lbs. and 8 lbs. act at an angle of 90 degrees to each other?

a) 8 lbs

b) $\sqrt{73}$ lbs

c) $2\sqrt{6}$ lbs

d) 2.66 lbs

Answer: b

Unit 4 – Practice Exam Questions

Unit Three – Topics 1 and 2 – Data and Central Tendency

Use the following data to answer questions 1-5

2, 12, 5, 16, 16, 1, 32

Question 1

What is the mode?

a) 12

b) There are several modes.

c) There is no mode.

d) 16

Answer: d

Question 2

What is the median?

a) 12

b) 16

c) Depends on quartile.

d) 6

Answer: a

Question 3

What is the mean?

a) 32

b) 16

c) 12

d) 15.5

Answer: c

Unit 4 – Practice Exam Questions

Unit Three – Topics 1 and 2 – Data and Central Tendency

Question 4

What is the range of the data?

a) 16

b) 1 to 32

c) 28

d) 64

Answer: b

Question 5

In a box and whisker plot of the data, where would the right end of the box fall?

a) 16

b) 24.5

c) 21.3

d) 8.5

Answer: c

Question 6

How much of the data is contained inside the box in a box and whisker plot?

a) The first quartile.

b) The second quartile.

c) Half of the data.

d) Two thirds of the data.

Answer: c

Unit 4 – Practice Exam Questions

Unit Three – Topics 1 and 2 – Data and Central Tendency

Question 7

When a value is added to a list that is much greater than the other values in the list, what will happen to the mean?

a) Nothing.

b) It will increase.

c) It will decrease.

d) It will become a mode.

Answer: b

Question 8

A new article reports that 30% of the people in a community are illiterate. What can you conclude about 70% of the people in this community?

a) They are younger.

b) They had more schooling.

c) They can read.

d) They had more desire to learn how to read.

Answer: c

Unit 4 – Practice Exam Questions

Unit 2 – Topic 3 – Probability

Question 1

What is the probability of getting 2 or 4 when you toss a die?

a) 1

b) 1/2

c) 1/3

d) 1/6

Answer: c

Question 2

When a card is dealt from a well shuffled deck of 52 cards, what is the sample space?

a) 1/52

b) 52

c) 1

d) 0

Answer: b

Question 3

What is the probability of getting an even number and a 3 on the toss of a die?

a) 1/3

b) 1/2

c) 1/6

d) 0

Answer: d

Unit 4 – Practice Exam Questions

Unit 2 – Topic 3 – Probability

Question 4

A fair spinner has 12 numbers on it. How would you use the Monte Carlo method to find the probability of getting an 8?

a) Divide the number of outcomes = 8 by 12.

b) Spin many times, record the results, count the times that 8 came up then divide by 12.

c) Find the union of 8 and 12.

d) Find the intersection of 8 and 12.

Answer: b

Question 5

What is the definition of probability?

a) Number of events divided by number of outcomes possible.

b) Number of sample space divided by number of events.

c) Favourable outcomes divided by possible outcomes.

d) Union of favourable events and the sample space.

Answer: c

Question 6

What is the probability of getting a face card in the suit of hearts from a card dealt from a well shuffled deck of 52 cards (4 suits, 3 face cards in each suit)?

a) 3/52

b) 6/52

c) 13/52

d) 4/52

Answer: a

Unit 4 – Practice Exam Questions

Unit 2 – Topic 3 – Probability

Question 7

At the Thebacha Campus of Aurora College 40% of the students are boarding away from home, 60% of the students are female and 30% are female and board away from home. What is the probability that a student chosen at random will be either female or a student who boards away from home?

a) .5

b) .6

c) .7

d) 3.5

Answer: c

Question 8

Five balls have numbers 1,2,3,4,5 painted on them and are placed in a jar. What is the probability of picking either 3 or 5?

a) 5/2

b) 5/7

c) 2/5

d) 1/5 - .1

Answer: c

Question 9

What is the probability of rolling two dice and getting a sum equal to 3?

a) 1/6

b) 1/18

c) 1/36

d) 3/36

Answer: b

Unit 5

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Appendix A

Math – Module 5 covers the following core requirements from the Alberta trades entrance curriculum:

Section 4 – Variables and Equations

E. Exponential, Logarithmic And Trigonometric Equations And Identities

1. Solve exponential equations having bases that are powers of one another.
2. Solve and verify exponential and logarithmic equations and identities.
3. Distinguish between degree and radian measure and solve problems using both.
4. Determine the exact and approximate values of trigonometric ratios for any multiples of 0, 30, 45, 60, and 90 degrees.
5. Solve first and second degree trigonometric equations over a domain of length 2pi algebraically and graphically.
6. Determine the general solution to trigonometric equations where the domain is the set of real numbers.
7. Verify trigonometric identities: numerically for any particular case; algebraically for general cases; and graphically.
8. Use sum difference and double angle identities for sine and cosine to verify and simplify trigonometric expressions.

Section 5 – Relations and Functions

A. Exponential And Logarithmic Functions Using Appropriate Technology

1. Graph and analyze an exponential function using technology.
2. Model, graph and apply exponential functions to solve problems.
3. Change functions from exponential form to logarithmic form and vice versa.
4. Use logarithms to model practical problems.
5. Explain the relationship between the laws of logarithms and the laws of exponents.
6. Graph and analyze logarithmic functions with and without technology.

B. Trigonometric Functions Using Appropriate Technology

1. Describe the three primary trigonometric functions as circular functions with reference to the unit circle and an angle in standard position.
2. Draw using technology, sketch and analyze graphs of sine, cosine and tangent functions for amplitude, period, domain and range, asymptotes, behaviour under transformations.
3. Draw using technology, sketch and analyze graphs of secant, cosecant and cotangent functions for amplitude, period, domain and range, asymptotes, behaviour under transformations.
4. Use trigonometric functions to model and solve problems.

Section 7 – 3D Objects and 2D Shapes

E. Problem Solving Using Polygons And Vectors

1. Use and give 2D and 3D examples of vector terminology and notation including: vector direction & magnitude, scalar, unit vector, collinear vectors, opposite vectors, parallel vectors, resultant vectors.
2. Assign meaning to the multiplication of a vector by a scalar.
3. Perform vector addition and subtraction using triangle or parallelogram methods.
4. Determine the magnitude and direction of a resultant vector, using triangle, parallelogram or component methods.
5. Use vector diagrams and trigonometry to analyze and solve practical problems in 2D and 3D.

Section 9 – Statistics and Probability

A. Data Analysis

1. Formulate questions for investigation, from a real-world context.
2. Select, defend and use appropriate methods of collecting data; designing and using questionnaires, interviews, experiments, research.
3. Describe issues to be considered when collecting data.
4. Display data by hand or by computer in a variety of ways, including circle graphs.
5. Read and interpret graphs.
6. Determine measures of central tendency for a set of data: mode, median, mean.
7. Determine measures of the distribution of a set of data: range, extremes, gaps and clusters, quartiles.
8. Interpolate from data to make predictions.

B. Plan For The Collection, Display And Analysis Of Data

1. Formulate questions for investigation, using existing data.
2. Select, use and defend appropriate methods of collecting data; designing and using surveys, research, using electronic media.
3. Display data by hand or by computer in a variety of ways, including box and whisker plots.

C. Evaluate And Use Measures

1. Determine and use the most appropriate measure of central tendency in a given context.
2. Describe the variability of data sets, using such techniques as range, and box and whisker plots.
3. Construct sets of data given measures of central tendency and variability.
4. Determine the effect on the mean, median and/or mode when a constant is added or subtracted from each value; each value is multiplied or divided by the same constant; and a significantly different value is included.

D. Collect and Analyse Experimental Data

1. Design, conduct and report on an experiment to investigate a relationship between two variables.
2. Create scatter plots for discrete and continuous variables.
3. Interpret a scatter plot to determine if there is an apparent relationship.
4. Determine the lines of best fit from a scatter plot for an apparent linear relationship, by inspection and using technology (equations are not expected).
5. Draw and justify conclusions from the line of best fit.
6. Assess the strengths, weaknesses and biases of samples and data collection methods.
7. Critique ways in which statistical information and conclusions are presented by the media and other sources.

E. Chance And Uncertainty

1. Use a table to identify all possible outcomes of two independent events.
2. Create and solve problems, using the numerical definition of probability as favourable outcomes divided by possible outcomes.
3. Use the Monte Carlo simulation method to solve probability problems.

F. Theoretical And Experimental Probability

1. Use computer or other simulations to solve probability and data collection problems.
2. Recognize that if n events are equally likely the probability of any one of them occurring is 1/n.
3. Determine the probability of two independent events where the combined sample space has 52 or fewer elements.
4. Predict population characteristics from sample data.

G. Complex Problems Using Probability And Statistics

Outcome: Explain the use of probability and statistics in the solution of complex problems. (5)

1. Recognize that decisions based on probability may be a combination of theoretical calculations, experimental results and subjective judgments.
2. Demonstrate an understanding of the role of probability and statistics in society.
3. Solve problems involving the probability of independent events.